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Note

Conjectures on wirelength of hypercube into cylinder and torus

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In the paper of Manuel et al. [8] the minimum wirelength of embedding hypercube into cylinder and torus were given as conjectures. In a recent paper of Rajan et al. [11] these conjectures have been proved. But there are logical flaws in the proof of lower bound of two conjectures and a constructional flaw in the upper bound of wirelength of hypercube into cylinder. In this paper we correct the constructional flaw and could not fix the logical flaws.

1. Introduction

A graph G is defined as a pair $G = (V, E)$, where $V = V(G)$ is a non-empty set of vertices and $E = E(G)$ is a set of edges. Embeddings are of great importance in the applications of parallel computing. Every parallel application has its intrinsic communication pattern. The communication pattern graph is mapped onto the topology of multiprocessor structures so that the corresponding application can be executed with minimal communication overhead.

Given a guest graph G and a host graph H, a graph embedding of G into H is an ordered pair $\prec f, P_f \succ$, where f is an injective map from $V(G)$ to $V(H)$ and P_f is also an injective map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$ [10,12]. An edge congestion of an embedding $\prec f, P_f \succ$ of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_{\prec f, P_f \succ}(e)$ denote the number of edges (u,v) of G such that the path $P_f(u,v)$ contains the edge e in H [9]. In other words, $EC_{\prec f, P_f \succ}(e) = \left|\left\{(u,v) \in E(G) : e \in E(P_f(u,v))\right\}\right|$.

The wirelength [7,9] of an embedding $\prec f, P_f \succ$ of G into H is given by

$$WL_{\prec f, P_f \succ}(G,H) = \sum_{(u,v)\in E(G)} \left|P_f(u,v)\right| = \sum_{e \in E(H)} EC_{\prec f, P_f \succ}(e).$$

The minimum wirelength of G into H is defined as $WL(G,H) = \min WL_{\prec f, P_f \succ}(G,H)$ where the minimum is taken over all embeddings $\prec f, P_f \succ$ of G into H.

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The isoperimetric problem [7] has been used as a powerful tool in the computation of minimum wirelength of graph embeddings. The following two versions of the edge isoperimetric problems (*NP*-complete) of a graph have been considered in the literature [1,4].

Version 1: Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,\, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $\theta_G(m) = |\theta_G(A)|$.

Version 2 (Maximum subgraph problem): Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V,\, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $I_G(m) = |I_G(A)|$.

We call such a set A optimal. Clearly, if a subset of vertices is optimal with respect to Version 1, then its complement is also an optimal set. However, it is not true for Version 2 in general, although this is indeed the case if the graph is regular [1].

In 1964, Harper computed the minimum wirelength of embedding hypercube into path [6]. In 2009, Manuel et al. obtained the minimum wirelength of hypercube into grid using the following lemmas [9]. In the sequel, the notation of embedding $\prec f, P_f \succ$ is represented shortly as f.

Lemma 1 *(Congestion lemma). (See [9].) Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge $(a,b) \in G_i$, $i = 1, 2$, $P_f(a,b)$ has no edges in S.*
(ii) *For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a,b)$ has exactly one edge in S.*
(iii) *G_1 is a maximum subgraph on $|V(G_1)|$ vertices.*
Then $EC_f(S)$ is minimum and $EC_f(S) = \sum_{e\in S} EC_f(e) = r\,|V(G_1)| - 2\,|E(G_1)|$.

Lemma 2 *(Partition lemma). (See [9].) Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

2. Two conjectures

The minimum wirelength of hypercube Q^r into cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$ and torus $C_{2^{r_1}} \times C_{2^{r_2}}$ have been proposed as conjectures [8] and given below.

Conjecture 1. *$WL(Q^r, C_{2^{r_1}} \times P_{2^{r_2}}) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \times 2^{2r_1-3} - 2^{r_1-1})$, where $r_1 + r_2 = r$, $r_1 \le r_2$.*

Conjecture 2. *$WL(Q^r, C_{2^{r_1}} \times C_{2^{r_2}}) = 2^{r_1}(3 \times 2^{2r_2-3} - 2^{r_2-1}) + 2^{r_2}(3 \times 2^{2r_1-3} - 2^{r_1-1})$, where $r_1 + r_2 = r$, $r_1 \le r_2$.*

In the recent paper [11], the above conjectures have been proved. The main idea of the paper was to use congestion lemma and partition lemma on certain edges of cylinder in order to get the minimum congestion and to apply various other procedures on the rest of the edges, but there are some flaws in the procedures.

Lemma 3. *(See Lemma 2.3 of [11].) Let G be the r-dimensional hypercube Q^r and H be the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, where $r_1 + r_2 = r$, $r_1 \le r_2$. Let S_j be an edge cut of H consisting of edges between the columns j and $j+1$ of H, $1 \le j \le 2^{r_2} - 1$. Then $\sum_{j=1}^{2^{r_2}-1} EC(S_j) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1})$.*

There is no clarity in the above lemma. For instance, the notation $EC(S_j)$ is nowhere defined in the paper [11]. The lemma is supposed to be:

Lemma 4. *Let $f : Q^r \to C_{2^{r_1}} \times P_{2^{r_2}}$ be any embedding, $r_1 + r_2 = r$, $r_1 \le r_2$. Let S_j be an edge cut of $C_{2^{r_1}} \times P_{2^{r_2}}$ consisting of edges between the columns j and $j+1$, $1 \le j \le 2^{r_2} - 1$. Then $\sum_{j=1}^{2^{r_2}-1} EC_f(S_j) \ge 2^{r_1}(2^{2r_2-1} - 2^{r_2-1})$.*