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Excuse me! or the courteous theatregoers' problem ☆

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ABSTRACT

Consider a theatre consisting of m rows each containing n seats. Theatregoers enter the theatre along aisles and pick a row which they enter along one of its two entrances so as to occupy a seat. Assume they select their seats uniformly and independently at random among the empty ones. A row of seats is narrow and an occupant who is already occupying a seat is blocking passage to new incoming theatregoers. As a consequence, occupying a specific seat depends on the courtesy of theatregoers and their willingness to get up so as to create free space that will allow passage to others. Thus, courtesy facilitates and may well increase the overall seat occupancy of the theatre. We say a theatregoer is *courteous* if (s)he will get up to let others pass. Otherwise, the theatregoer is *selfish*. A set of theatregoers is *courteous with probability p* (or *p-courteous*, for short) if each theatregoer in the set is courteous with probability p, randomly and independently. It is assumed that the behaviour of a theatregoer does not change during the occupancy of the row. Thus, $p = 1$ represents the case where all theatregoers are courteous and $p = 0$ when they are all selfish.
In this paper, we are interested in the following question: what is the expected number of occupied seats as a function of the total number of seats in a theatre, n, and the probability that a theatregoer is courteous, p? We study and analyze interesting variants of this problem reflecting behaviour of the theatregoers as entirely selfish, and p-courteous for a row of seats with one or two entrances and as a consequence for a theatre with m rows of seats with multiple aisles. We also consider the case where seats in a row are chosen according to the geometric distribution and the Zipf distribution (as opposed to the uniform distribution) and provide bounds on the occupancy of a row (and thus the theatre) in each case. Finally, we propose several open problems for other seating probability distributions and theatre seating arrangements.

1. Introduction

A group of Greek tourists is vacationing on the island of Lipari and they find out that the latest release of their favourite playwright is playing at the local theatre (see Fig. 5), *Ecclesiazusae* (or Assemblywomen) by Aristophanes, a big winner at last year's (391 BC) Festival of Dionysus. Seating at the theatre is open (i.e., the seats are chosen by the audience members

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as they enter). The question arises as to whether they will be able to find seats. As it turns out this depends upon just how courteous the other theatregoers are that night.

Consider a theatre with m rows containing n seats each. Theatregoers enter the theatre along aisles, choose a row, and enter it from one of its ends, wishing to occupy a seat. They select their seat in the row uniformly and independently at random among the empty ones. The rows of seats are narrow and if an already sitting theatregoer is not willing to get up then s(he) blocks passage to the selected seat and the incoming theatregoer is forced to select a seat among unoccupied seats between the row entrance and the theatregoer who refuses to budge. Thus, the selection and overall occupancy of seats depends on the courtesy of sitting theatregoers, i.e., their willingness to get up so as to create free space that will allow other theatregoers go by.

An impolite theatregoer, i.e., one that never gets up from a position s(he) already occupies, is referred to as *selfish* theatregoer. Polite theatregoers (those that will get up to let someone pass) are referred to as *courteous*. On a given evening we expect some fraction of the audience to be selfish and the remainder to be courteous. We say a set of theatregoers is *p-courteous* if each individual in the set is courteous with probability p and selfish with probability $1-p$. We assume that the status of a theatregoer (i.e., selfish or courteous) is independent of the other theatregoers and it remains the same throughout the occupancy of the row. Furthermore, theatregoers select a vacant seat uniformly at random. They enter a row from one end and inquire ("Excuse me"), if necessary, whether an already sitting theatregoer is courteous enough to let him/her go by and occupy the seat selected. If a selfish theatregoer is encountered, a seat is selected at random among the available unoccupied ones, should any exist. We are interested in the following question:

> What is the expected number of seats occupied by theatregoers when all new seats are blocked, as a function of the total number of seats and the theatregoers' probability p of being courteous?

We first study the problem on a single row with either one entrance or two. For the case $p=1$ it is easy to see that the row will be fully occupied when the process finishes. We show that for $p=0$ (i.e., all theatregoers are selfish) the expected number of occupied seats is only $2\ln n + O(1)$ for a row with two entrances. Surprisingly, for any fixed $p<1$ we show that this is only improved by essentially a constant factor of $\frac{1}{1-p}$.

Some may argue that the assumption of choosing seats uniformly at random is somewhat unrealistic. People choose their seats for a number of reasons (sight lines, privacy, etc.) which may result in a nonuniform occupancy pattern. A natural tendency would be to choose seats closer to the centre of the theatre to achieve better viewing. We attempt to model this with seat choices made via the geometric distribution with a strong bias towards the centre seat for the central section of the theatre and for the aisle seat for sections on the sides of the theatre. The results here are more extreme, in that for p constant, we expect only a constant number of seats to be occupied when there is a bias towards the entrance of a row while we expect at least half the row to be filled when the bias is away from the entrance. In a further attempt to make the model more realistic we consider the Zipf distribution [1] on the seat choices, as this distribution often arises when considering the cumulative decisions of a group of humans (though not necessarily Greeks). We show that under this distribution when theatregoers are biased towards the entrance to a row, the number of occupied seats is $\Theta(\ln\ln n)$ while if the bias is towards the centre of the row the number is $\Theta(\ln^2 n)$. If we assume that theatregoers proceed to another row if their initial choice is blocked it is easy to use our results for single rows with one and two entrances to derive bounds on the total number of seats occupied in a theatre with multiple rows and aisles.

1.1. Related work

Motivation for seating arrangement problems comes from polymer chemistry and statistical physics in [2,3] (see also [4, Chapter 19] for a related discussion). In particular, the number and size of random independent sets on grids (and other graphs) is of great interest in statistical physics for analyzing *hard* particles in lattices satisfying the exclusion rule, i.e., if a vertex of a lattice is occupied by a particle its neighbours must be vacant, and have been studied extensively both in statistical physics and combinatorics [5–9].

Related to this is the "unfriendly seating" arrangement problem which was posed by Freedman and Shepp [10]: Assume there are n seats in a row at a luncheonette and people sit down one at a time at random. Given that they are unfriendly and never sit next to one another, what is the expected number of persons to sit down, assuming no moving is allowed? The resulting density has been studied in [10–12] for a $1\times n$ lattice and in [13] for the $2\times n$ and other lattices. See also [14] for a related application to privacy.

Another related problem considers the following natural process for generating a maximal independent set of a graph [15]. Randomly choose a node and place it in the independent set. Remove the node and all its neighbours from the graph. Repeat this process until no nodes remain. It is of interest to analyze the expected size of the resulting maximal independent set. For investigations on a similar process for generating maximal matchings the reader is referred to [16,17].

1.2. Outline and results of the paper

We consider the above problem for the case of a row that has one entrance and the case with two entrances. We develop closed form formulas, or almost tight bounds up to multiplicative constants, for the expected number of occupied