Theoretical Computer Science 580 (2015) 14–27

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Graphical limit sets for general cellular automata

Johannes Müller [a,b,*], Hengrui Jiang [a]

[a] TU München, Centre for Mathematical Sciences, Boltzmannstr. 3, D-85747 Garching, Germany
[b] Helmholtz Center Munich, Institute for Computational Biology, Ingolstädter Landstr. 1, D-85764 Neuherberg, Germany

ARTICLE INFO

Article history:
Received 24 April 2014
Received in revised form 22 January 2015
Accepted 12 February 2015
Available online 3 March 2015
Communicated by J. Kari

Keywords:
Cellular automata
Graphical limit sets

ABSTRACT

The existing theory of graphical limit sets for cellular automata relies on algebraic structures and applies only to certain classes of cellular automata that possess this structure. We extend this theory to general cellular automata using topological methods. The starting point is the observation that the rescaled space-time diagrams, intersected with an appropriately chosen compact set, form sequences in a compact, metric space. They necessarily possess converging subsequences. In the present paper we define graphical limit sets as the collection of the accumulation points. The main result is that for a large class of cellular automata the graphical limit set defined in this way carries a group structure, which is either the trivial group consisting of one element only, or is homeomorphic to S^1. The well known self-similar, graphical limit sets are representatives of the second class.

1. Introduction

A central problem in the theory of cellular automata is the classification of these objects in sensible subsets. Cellular automata are – similarly to e.g. finitely generated groups, or partial differential equations – much too diverse to allow for many strong theorems covering all of them. The Curtis–Lyndon–Hedlund theorem [5,1] may be one of the very few examples for such general theorems. The general feeling is that an appropriate classification yields subgroups that can be well understood. Since the seminal work of Wolfram [19], where a purely phenomenological classification by visual inspection has been proposed, there have been many attempts to find formal justifications and methods to introduce classification schemes. Examples of these approaches are topological dynamical systems (Hurley classification [6,7]), or formal grammars (Gilman classification [2]) or continuity (Kůrka classification [9]). We aim to contribute some ideas that are perhaps close to Wolfram's original, phenomenological concept. This concept has been formalised by Wilson for some cellular automata. The space-time diagrams of Wolfram automata yield in some cases structures resembling self-similar patterns. Taking an appropriate limit, Willson [14,15] has been able to show that the space-time patterns of certain cellular automata indeed tend to self-similar sets. This observation gained a lot of attention [17,16,12,11,10]. The core assumption has always been the existence of certain algebraic properties of the cellular automaton. Recent developments attempt to relax the conditions, using more general algebraic structures [4].

In the present work, we propose to define graphical limit sets in a purely topological way. No algebraic structures are required. We simply use the fact, that the set of non-void, compact sets which are uniformly bounded together with the Hausdorff metric form a compact metric space. This fact indicates that the sequence of graphical representations of

* Corresponding author at: TU München, Centre for Mathematical Sciences, Boltzmannstr. 3, D-85747 Garching, Germany.
E-mail addresses: johannes.mueller@mytum.de (J. Müller), Hengrui.jiang@googlemail.com (H. Jiang).

http://dx.doi.org/10.1016/j.tcs.2015.02.022
0304-3975/

space-time patterns evolving during simulations of cellular automata have accumulation points. The graphical limit set of a cellular automaton is defined as the collection of these accumulation points. We investigate the structure of this set.

2. Definition of graphical limit sets

We recall that a cellular automaton is defined by a tuple, such as (Γ, U_0, E, f_0), where Γ is a finitely generated Abelian group (e.g., $\mathbb{Z}$), $U_0 \subset \Gamma$ is a finite set denoting the neighborhood of the neutral element (called “origin” in the present context) 0 in Γ, E is a finite set of local states. Let E^{U_0} (resp. E^Γ) denote the set of all maps from U_0 to E, $\{g : U_0 \to E\}$ (resp. $\{g : \Gamma \to E\}$); the elements in E^{U_0} (E^Γ) are called configurations or states. $f_0 : E^{U_0} \to E$ is the local function. The global function of the cellular automaton is defined by $f : E^\Gamma \to E^\Gamma$, $u \mapsto f(u)$ where $f(u)(z) = f_0(u|_{z+U_0})$.

Definition 2.1. Given a cellular automaton (Γ, U_0, E, f_0) with $\Gamma = \mathbb{Z}$, $E = \{0, \ldots, Q\}$. The state $\bar{0} \in E^{U_0}$ denotes $\bar{0}(x) = 0$ for all $x \in U_0$. If $f_0(\bar{0}) = 0$, then 0 is called quiescent or resting state. The support of $u_0 \in E^\Gamma$ is defined by $\mathrm{supp}(u_0) = \{x \in \Gamma \mid u_0(x) \neq 0\}$. Denote by $(E^\Gamma)_c$ the states with finite (or compact) support. We define

$$P(u_0) = \{(z,t) \in \mathbb{Z} \times \mathbb{N}_0 \mid f^t(u_0)(z) \neq 0\}.$$

Notation: For $a, b \in \mathbb{Z}$, we denote by $[a, b] = \{a, \ldots, b\} \subset \mathbb{Z}$; if $[a, b] \subset \mathbb{R}$ is meant, this becomes clear from the context.

Let $U_0 \subset [-d, d]$ and $\mathrm{supp}(u_0) \subset [-K, K]$. Assume that the cellular automaton possesses a quiescent state 0. Then,

$$P(u_0) \subseteq \bigcup_{t \in \mathbb{N}} [-(dt + K), (dt + K)] \times \{t\} =: \Delta.$$

In any case, also without the assumption of a quiescent state, the only part of $P(u_0)$ influenced by the non-trivial information stored in the initial state u_0 is contained in Δ. Therefore, Δ is also called the “light-cone” of the cellular automaton – the information contained in u_0 only spreads within this set. Outside this set, the automaton cycles through states that are constant in space; to be more precise, we find the same pattern that can be observed if we start with the initial value $\bar{0} \in E^\Gamma$ (the state that is identically zero). Rescaling $P(u_0)$ yields the definition of the graphical representation.

Definition 2.2 *(Recall notation of Definition 2.1).* Let $I = [-1/2, 1/2] \times [0, 1]$, and for $(z, t) \in \mathbb{Z}^2$ define $I_{z,t} = (z, t) + I$. Let henceforth the set J be defined as

$$J = [-(d + K + 1), d + K + 1] \times [0, 1].$$

The rescaled graphical representation of the space-time pattern (or simply the space-time pattern) is defined as

$$F_n(u_0) = \left(\frac{1}{n} \bigcup_{(z,t) \in P(u_0)} I_{z,t} \right) \cap J.$$

Remark 2.3. Due to the scaling by $1/n$, $F_n(u_0)$ only takes into account the first $n - 1$ time steps. The region of $F_n(u_0)$ influenced by the non-zero pattern in the initial condition of the initial state u_0 is contained in

$$\Delta_n(u_0) = \bigcup_{m \in [0, n-1]} [-(dm + K + 1/2)/n, (dm + K + 1/2)/n] \times [m/n, (m+1)/n] \subset J.$$

Outside of $\Delta_n(u_0)$, the set $F_n(u_0)$ is either empty or solid, or consists of horizontal strips. We will use this fact later. Let us denote the part of J that carries information for $n \to \infty$ by Δ_∞, i.e. $\Delta_\infty = \{(x, y) \mid -d\,y \le x \le d\,y\}$.

We aim at graphical limit sets, which means that we aim at an understanding of the convergence of subsequences of the sequence $(F_n(u_0))_{n \in \mathbb{N}}$. We recall the definitions of Hausdorff metric and Kuratowski convergence (see also [8]).

Definition 2.4. Let A, B be compact sets in $\mathbb{R}^n$. If either A or B is empty, define $d_H(A, B) = \infty$. Else, define

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid B \subseteq A_\varepsilon, \text{ and } A \subseteq B_\varepsilon\},$$

where $A_\varepsilon = \{x \mid \exists y \in A : \|x - y\| < \varepsilon\}$. $d_H(.,.)$ is called Hausdorff metric.

The Hausdorff metric introduces a topology on the set of all compact subsets of $\mathbb{R}^n$. A sequence of sets A_n converges to a set B, if $d_H(A_n, B) \to 0$.