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Reachability and recurrence in a modular generalization of annihilating random walks (and lights-out games) to hypergraphs

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ABSTRACT

We study a discrete asynchronous dynamical system on hypergraphs that can be regarded as a natural extension of annihilating walks along two directions: first, the interaction topology is a hypergraph; second, the "number of particles" at a vertex of the hypergraph is an element of a finite ring $\mathbf{Z}_p$ of integers modulo an odd number $p \geq 3$. Equivalently particles move on a hypergraph, with a moving particle at a vertex being replaced by one indistinguishable copy at each neighbor in a given hyperedge; particles at a vertex collectively annihilate when their number reaches p.

The boolean version of this system arose in earlier work [22] motivated by the statistical physics of social balance [3,2], generalizes certain lights-out games [31] to finite fields and also has some applications to the complexity of local search procedures [23].

Our result shows that under a liberal sufficient condition on the nature of the interaction hypergraph there exists a polynomial time algorithm (based on linear algebra over $\mathbf{Z}_p$) for deciding reachability and recurrence of this dynamical system. Interestingly, we provide a counterexample that shows that this connection does *not* extend to all graphs.

1. Introduction

Interacting particle systems [25] are discrete dynamical systems, naturally related to cellular automata [18,13], that have seen extensive study in the Statistical Physics of Complex Systems. While they are most naturally studied on lattices, extensions to general graphs are possible. Such extensions have recently found many applications to social dynamics, particularly as *opinion formation models* (see [8] for a recent survey). In particular, the most popular interacting particle systems, the voter and antivoter model and (by duality) annihilating and coalescing random walks have also been studied on a general graph [10,11,1].

Extensions to hypergraphs are also possible and relevant in a social context. For instance, Lanchier and Neufer [26] argue for the naturalness of such an extension and give a spatial version of Galam's majority model [14] via a majority voting rule. Motivated by behavioral voting experiments on networks [24], Chung and Tsiatas study [9] a voter model on hypergraphs. A final example comes from the Statistical Physics of social balance [3,2]. A dynamical adjustment process introduced in

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these papers naturally leads via duality [22] to an extension of annihilating random walks to hypergraphs. This extension can be specified as follows:

Definition 1 *(Annihilating random walks on hypergraphs).* Particles live on the vertices of a hypergraph. At each moment:

1. We chose a random vertex v containing a particle.
2. We chose a random hyperedge e that contains vertex v.
3. Vertices in e that contain a particle (including v) become empty. On the other hand empty vertices in e will afterwards contain a particle.

The process specified at Step 3 can be described intuitively in the following way: the particle P at vertex e spawns a number of descendents, one for each vertex $z \in e \setminus v$, then dies. The new particles may meet already a pre-existing particle at vertex z, in which case the two particles "collectively annihilate". This dynamics, studied in [22], is also naturally related as we found out after completing [22], to a classical problem in the area of combinatorial games, the theory of *lights-out games* [31]. We further discuss this connection in the next section.

The remarkable aspect of the extension (Definition 1) of annihilating random walks to hypergraphs lies in its "explosive" nature: on hyperedges one particle may give birth to more than one copy. Thus, unlike the graph case, the total number of particles is generally *not* a nonincreasing function.

The purpose of this paper is to study a modulo-p version of the dynamical system from [22], specifically, the following system:

Definition 2. Let $p \geq 2$ be an integer. A $\mathbf{Z}_p$*-annihilating walk on a hypergraph G* is defined as follows: each node v of G is initially endowed with a number $w(v) \in \mathbf{Z}_p$ (interpreted as number of *particles*).

The allowed moves are specified as follows: choose a node v such that $w(v) \neq 0$ and a hyperedge e containing v. Change the state of $w(v)$ to $w(v) - 1$. Also change the state of every node $u \neq v$, $u \in e$ to $w(u) + 1$.

In other words: a number of indistinguishable particles are initially placed at the vertices of G, each vertex holding from 0 to $p - 1$ particles. At each step we choose a vertex v containing at least one particle and a hyperedge containing v. We delete one particle at v and add one particle at every vertex $w \neq v \in e$. If the number of particles at some w reaches p, these p particles are removed from w (they "collectively annihilate").

We are mainly interested in the complexity of the following two problems:

Definition 3 *(Reachability).* Given hypergraph $G = (E, V)$ and states $w_1, w_2 \in \mathbf{Z}_p^V$, decide whether w_2 is reachable from w_1.

Definition 4 *(Recurrence).* Given hypergraph $G = (E, V)$ and states $w_1, w_2 \in \mathbf{Z}_p^V$, decide whether w_2 is reachable from any state $w_3 \in \mathbf{Z}_p^V$ reachable from w_1.

Of course, reachability and recurrence are fundamental prerequisites for studying the *random* version of this dynamical system as a finite-state Markov chains, the problem that was the original motivation of our research.

There are simple algorithms that put the complexity of these two problems above in the complexity classes PSPACE and EXPSPACE, respectively: for REACHABILITY we simply consider reachability in the (exponentially large) state space directed graph S with vertex set $\mathbf{Z}_p^V$. For RECURRENCE we combine enumeration of all vertices w_3 reachable from w_1 (via breadth first search) with testing reachability of w_2 from w_3.

The main purpose of this paper is to show that under a quite liberal sufficient condition on the nature of underlying hypergraph reachability and recurrence questions for $\mathbf{Z}_p$-annihilating walks can be decided in polynomial time (actually they belong to the apparently weaker class Mod_p-L [4], but we won't discuss this issue here any further), by solving a certain system of linear equations over Z_p.

Of course, the above result is not entirely surprising, as it comes in an established line of applications of linear algebra to reachability problems in lights-out games (see [29] for a discussion and list of references). On the other hand, as discussed in the next section, the class of moves we allow is more restricted than that in the models in [29], and it was only recently shown [17] that in certain cases this restriction does not matter (we refer to the next section for a full discussion). We provide a counterexample that, interestingly, shows that our result is not generally valid if we eliminate the sufficient condition.

Throughout the paper we will assume that $p \geq 3$ is an odd number.

2. Related work

As mentioned in the introduction, the dynamics studied in [22] is a generalization to hypergraphs of a variant of the *lights out* (σ)*-game* [31], a problem that has seen significant investigation. The version we considered in [22] is the apparently more constrained *lit-only* σ^+*-game*: