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On the Cartesian skeleton and the factorization of the strong product of digraphs

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ABSTRACT

The three standard products (the Cartesian, the direct and the strong product) of undirected graphs have been repeatedly studied, unique prime factor decomposition (PFD) is known and polynomial time algorithms have been established for determining the prime factors.

For directed graphs, unique PFD results with respect to the standard products are known. However, there is, until now, no known algorithm to compute the PFD of directed graphs with respect to the direct and the strong product in general. In this contribution, we focus on the algorithmic aspects for determining the PFD of directed graphs with respect to the strong product. Essential for computing the prime factors is the construction of a so-called Cartesian skeleton. This article introduces the notion of the Cartesian skeleton of directed graphs as a generalization of the Cartesian skeleton of undirected graphs. We provide new, fast and transparent algorithms for its construction. It leads to the first polynomial-time algorithm for determining the PFD of arbitrary, finite connected digraphs with respect to the strong product.

1. Introduction

Graphs and in particular graph products arise in a variety of different contexts, from computer science [1,22] to theoretical biology [28,30], computational engineering [23–25] or just as natural structures in discrete mathematics [9,18].

We assume the reader is quite familiar with the three standard graph products. Nonetheless, we collect the definitions and some main ideas in this introduction. Given (directed) graphs H and K, there are three standard graph products, the Cartesian product $H \square K$, the direct product $H \times K$ and, the strong product $H \boxtimes K$. Each has as a vertex set the Cartesian product $V(H) \times V(K)$ of sets. Two vertices $(h_1, k_1), (h_2, k_2) \in V(H) \times V(K)$ are adjacent in $H \boxtimes K$ if one of the following conditions is satisfied:

$$\text{(i)}\quad h_1h_2 \in E(H) \quad\text{and}\quad k_1 = k_2 \qquad \text{(ii)}\quad h_1 = h_2 \quad\text{and}\quad k_1k_2 \in E(K) \qquad \text{(iii)}\quad h_1h_2 \in E(H) \quad\text{and}\quad k_1k_2 \in E(K).$$

In the Cartesian product $H \square K$ vertices are only adjacent if they satisfy (i) or (ii). Consequently, the edges of a strong product that satisfy (i) or (ii) are called *Cartesian*, the others *non-Cartesian*. In the direct product $H \times K$ vertices are adjacent if they fulfill (iii). See Fig. 1 for examples. In what follows, we assume that $\odot \in \{\square, \times, \boxtimes\}$. The one-vertex complete graph

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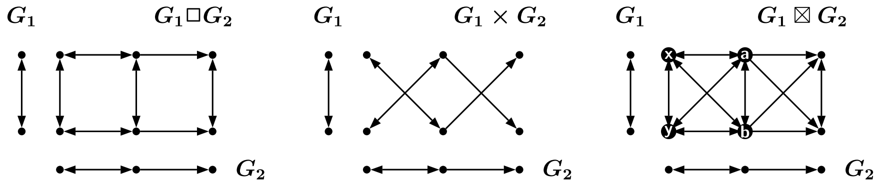

Fig. 1. The three standard products of directed graphs.

K_1 serves as unit $U_\boxtimes$ for the strong and unit $U_\Box$ for the Cartesian product, while the one-vertex complete graph $\mathcal{L}(K_1)$ with loop is the unit $U_\times$ for the direct product, as $U_\odot \odot G = G$ for all graphs G. A (directed) graph is called *prime* with respect to a particular product if it has at least two vertices and whenever it is expressed as a product of two factors, one of the factors is the unit for the product. An expression $G \cong G_1 \odot G_2 \odot \cdots \odot G_k$ with each G_i being prime w.r.t. $\odot$ is called *prime factor decomposition*, or *PFD* for short.

For undirected simple graphs, it is well-known that each of the three standard graph products, the Cartesian product [5,21,27,29], the direct product [17,26] and the strong product [2,6,26], satisfies the unique prime factor decomposition property under certain conditions, and there are polynomial-time algorithms to determine the prime factors. Two monographs cover the topic in substantial detail and serve as standard references [9,18].

For directed graphs, or digraphs for short, only partial results are known. Feigenbaum showed that the Cartesian product of digraphs satisfies the unique prime factorization property and provided a polynomial-time algorithm for its computation [3]. McKenzie proved that digraphs have a unique prime factor decomposition w.r.t. the direct product requiring strong conditions on connectedness [26]. This result was extended by Imrich and Klöckl [19,20]. The authors provided unique prime factorization theorems and a polynomial-time algorithm for the direct product of digraphs under relaxed connectivity, but additional so-called thinness conditions. The results of McKenzie also imply that the strong product of finite digraphs can be uniquely decomposed into prime factors [26]. Surprisingly, so far no general algorithm for determining the prime factors of the strong product of digraphs has been established.

In this contribution, we are concerned with the algorithmic aspect of the prime factor decomposition w.r.t. the strong product of digraphs. The key idea for the prime factorization of a strong product digraph $G = H \boxtimes K$ is the same as for undirected graphs: Since the strong product $G = H \boxtimes K$ contains as subgraph the Cartesian product $H \Box K$, the main goal is to find a subgraph $\mathbb{S}(G)$ of G that includes sufficient information about the Cartesian edges. Fast $\Box$-PFD algorithms on $\mathbb{S}(G)$ can then be used to determine the Cartesian prime factors of $\mathbb{S}(G)$ from which the strong prime factors of G can be inferred. Clearly, it would be sufficient, to remove the non-Cartesian edges of G, resulting in the subgraph $H \Box K$ of which one extracts the Cartesian factors K and H, and thus, the putative factors of G. However, this idea is not directly applicable. Firstly, the distinction of Cartesian and non-Cartesian edges requires that we already know the factorization $H \boxtimes K$ of G. Secondly, although $H \Box K \subseteq H \boxtimes K$ and H, K might be the strong prime factors of G, the $\Box$-PFD of $H \Box K$ can be finer if H or K are not prime w.r.t. the Cartesian product. Thirdly, the assignment of an edge being Cartesian or non-Cartesian is not unique, in general, even if the factorization $H \boxtimes K$ of G is known. By way of example, consider the strong product digraph in Fig. 1. There are two possibilities, either we set xa and yb as Cartesian implying that xb and ya are non-Cartesian, or vice versa, we set xb and ya as Cartesian implying that xa and yb are non-Cartesian. The reason for the non-unique assignment is the existence of automorphisms that interchange the vertices x and y, but fix all the others. This is possible because x and y have the same neighborhoods. However, when all vertices in a graph can be distinguished by their neighborhoods, which is the case for so-called "S-thin" graphs, one can uniquely determine Cartesian and non-Cartesian edges.

In this contribution, we aim at the construction of the Cartesian skeleton $\mathbb{S}(G)$ of directed graphs G, that is a subgraph of G that at least contains "sufficiently many" Cartesian edges. The skeleton $\mathbb{S}(G)$ is defined in terms of intersections and subset-relations of neighborhoods for S-thin graphs G. The prime factors of G w.r.t. the strong product are then constructed by utilizing the information of the Cartesian prime factors of $\mathbb{S}(G)$. This approach can easily be extended to graphs G that are not S-thin. We will show that the skeleton satisfies $\mathbb{S}(H \boxtimes K) = \mathbb{S}(H) \Box \mathbb{S}(K) \subseteq H \Box K$ for so-called S-thin digraphs. We prove that $\mathbb{S}(G)$ is a spanning subgraph of G and that it is connected whenever G is connected. We provide new, fast and transparent algorithms for its construction. Furthermore, we present the first polynomial-time algorithm for the computation of the PFD w.r.t. the strong product of arbitrary connected digraphs.

2. Preliminaries

2.1. Basic notation

A *digraph* $G = (V, E)$ is an ordered pair consisting of a set of vertices $V(G) = V$ and a set of ordered pairs $xy \in E(G) = E$, called (directed) edges or arcs. In what follows, we consider only simple digraphs with finite vertex and edge set. It is possible that both, xy and yx are contained in E. However, we only consider digraphs without loops, i.e., $xx \notin E$ for all $x \in V$. An *undirected graph* $G = (V, E)$ is an ordered pair consisting of a set of vertices $V(G) = V$ and a set of unordered pairs $\{x, y\} \in E(G) = E$. The *underlying undirected* graph of a *digraph* $G = (V, E)$ is the graph $U(G) = (V, F)$ with edge set $F = \{\{x, y\} \mid xy \in E \text{ or } yx \in E\}$. A digraph H is a *subgraph* of a digraph G, in symbols $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.