Theoretical Computer Science 560 (2014) 219–234

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Down the Borel hierarchy: Solving Muller games via safety games ☆

Daniel Neider [a], Roman Rabinovich [b,1], Martin Zimmermann [a,c,*,2]

[a] *Lehrstuhl für Informatik 7, RWTH Aachen University, 52056 Aachen, Germany*
[b] *Lehr- und Forschungsgebiet Mathematische Grundlagen der Informatik, RWTH Aachen University, 52056 Aachen, Germany*
[c] *Institute of Informatics, University of Warsaw, 02-097 Warsaw, Poland*

ARTICLE INFO

Available online 18 January 2014

Keywords:
Muller games
Safety games
Permissive strategies
Game reductions

ABSTRACT

We transform a Muller game with n vertices into a safety game with $(n!)^3$ vertices whose solution allows us to determine the winning regions of the Muller game and to compute a finite-state winning strategy for one player. This yields a novel antichain-based memory structure, a compositional solution algorithm, and a natural notion of permissive strategies for Muller games. Moreover, we generalize our construction by presenting a new type of game reduction from infinite games to safety games and show its applicability to several other winning conditions.

1. Introduction

Muller games are a source of interesting and challenging questions in the theory of infinite games. They are expressive enough to describe all ω-regular properties. Also, all winning conditions that depend only on the set of vertices visited infinitely often can trivially be reduced to Muller games. Hence, they subsume Büchi, co-Büchi, parity, Rabin, and Streett conditions. Furthermore, Muller games are not positionally determined, i.e., both players need memory to implement their winning strategies. In this work, we consider three aspects of Muller games: solution algorithms, memory structures, and quality measures for strategies.

To date, there are two main approaches to solve Muller games: direct algorithms and reductions. Examples for the first approach are Zielonka's recursive polynomial space algorithm [1] which is based on earlier work by McNaughton [2], and Horn's polynomial time algorithm for explicit Muller games [3]. The second approach is to reduce a Muller game to a parity game using Zielonka trees [4] or latest appearance records (LAR) [5].

In general, the number of memory states needed to win a Muller game is prohibitively large [4]. Hence, a natural task is to reduce this number (if possible) and to find new memory structures which may implement small winning strategies in subclasses of Muller games.

As for the third aspect, to the best of our knowledge there is no previous work on quality measures for strategies in Muller games. This is in contrast to other winning conditions. Recently, much attention is being paid to not just synthesize some winning strategy, but to find an optimal one according to a certain quality measure, e.g., waiting times in

☆ This work was supported by the projects *Games for Analysis and Synthesis of Interactive Computational Systems* (*GASICS*) and *Logic for Interaction* (*LINT*) of the *European Science Foundation* and by the European Union's Seventh Framework Programme (FP7/2007–2013) under grant agreement 239850 (SOSNA).

* Corresponding author.
E-mail addresses: neider@automata.rwth-aachen.de (D. Neider), roman.rabinovich@tu-berlin.de (R. Rabinovich), zimmermann@react.uni-saarland.de (M. Zimmermann).
[1] Present address: Logic and Semantics Research Group, Technische Universität Berlin, 10587 Berlin, Germany.
[2] Present address: Reactive Systems Group, Saarland University, 66123 Saarbrücken, Germany.

0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2014.01.017

request–response games [6] and their extensions [7], permissiveness in parity games [8,9], bounds in finitary games [10] and games with costs [11], and the use of weighted automata in quantitative synthesis [12,13].

Inspired by work of McNaughton [14], we present a framework to deal with all three issues. Our main contributions are a novel algorithm and a novel type of memory structure for Muller games. We also obtain a natural quality measure for strategies in Muller games and are able to extend the definition of permissiveness to Muller games.

While investigating the interest of Muller games for "casual living-room recreation" [14], McNaughton introduced scoring functions which describe the progress a player is making towards winning a play of the game: consider a Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{A}$ is the arena and $(\mathcal{F}_0, \mathcal{F}_1)$ is a partition of the set of strongly connected subsets in $\mathcal{A}$ used to determine the winner. Then, the score of a set F of vertices measures how often F has been visited completely since the last visit of a vertex not in F. Player i wins a play in the Muller game if and only if there is an $F \in \mathcal{F}_i$ such that the score of F tends to infinity while being reset only finitely often (a reset occurs whenever a vertex outside F is visited).

McNaughton proved the existence of strategies for the winning player that bound her opponent's scores by $|\mathcal{A}|!$ [14], if the play starts in her winning region. The characterization above implies that such a strategy is necessarily winning. The bound $|\mathcal{A}|!$ was subsequently improved to two (and shown to be tight) [15]. Since some score eventually reaches value three, the winning regions of a Muller game can therefore be determined by solving the reachability game on a finite tree in which a player wins if she is the first to reach a score of three.[3] However, it is cumbersome to obtain a winning strategy for the infinite-duration Muller game from a winning strategy for the finite-duration reachability game. The reason is that one has to carefully concatenate finite plays of the reachability game to an infinite play of the Muller game: reaching a score of three infinitely often does not prevent the opponent from visiting other vertices infinitely often.

Our contributions The ability to bound the losing player's scores can be seen as a safety condition as well. We use this to devise an algorithm to solve Muller games that computes both winning regions and a winning strategy for one player via a reduction to safety games. However, we do not obtain a winning strategy for the other player. In general, it is impossible to reduce a Muller game to a safety game whose solution yields winning strategies for both players, since safety conditions are on a lower level of the Borel hierarchy than Muller conditions.

Given a Muller game, we construct a safety game in which the scores of Player 1 are tracked (up to score three). Player 0 wins the safety game, if she can prevent Player 1 from ever reaching a score of three for every $F \in \mathcal{F}_1$. This allows us to compute the winning region of the Muller game by solving a safety game. Making use of the conjunctive nature of the winning condition of the safety game, we are also able to give a compositional algorithm for solving Muller games by solving the resulting safety game compositionally.

Furthermore, by exploiting the intrinsic structure of the safety game's arena we present an antichain-based memory structure for Muller games. Unlike the memory structures induced by Zielonka trees, which disregard the structure of the arena, and the ones induced by LARs, which disregard the structure of the winning condition $(\mathcal{F}_0, \mathcal{F}_1)$, our memory structure takes both directly into account: a simple arena or a simple winning condition should directly lead to a small memory. The other two memory structures only take one source of simplicity into account. Also, our memory implements the most general non-deterministic winning strategy among those that prevent the opponent from reaching a certain score in a Muller game. Thus, our framework allows us to extend the notion of permissiveness from positionally determined games to games that require memory.

Our idea of turning a Muller game into a safety game can be generalized to other types of winning conditions. We define a novel notion of reduction from infinite games to safety games which not only subsumes our construction but generalizes several constructions found in the literature. Based on work on small progress measures for solving parity games [16], Bernet, Janin, and Walukiewicz showed how to determine the winning regions in a parity game and a winning strategy for one player by reducing it to a safety game [8]. Furthermore, Schewe and Finkbeiner [17] as well as Filiot, Jin, and Raskin [18] used a translation from co-Büchi games to safety games in their work on bounded synthesis and LTL realizability, respectively. We present further examples and show that our reduction allows us to determine the winning region and a winning strategy for one player by solving a safety game. Thus, all these games can be solved by a new type of reduction and an algorithm for safety games. Our approach simplifies the winning condition of the game, even down the Borel hierarchy. However, this is offset by an increase in the size of the arena. Nevertheless, in the case of Muller games, our arena is only cubically larger than the arena constructed in the reduction to parity games. Furthermore, a safety game can be solved in linear time, while the question of whether there is a polynomial time algorithm for parity games is open.

2. Definitions

The power set of a set V is denoted by 2^V, and $\mathbb{N}$ denotes the set of non-negative integers. The prefix relation on words is denoted by $\sqsubseteq$. For $\rho \in V^\omega$ and $L \subseteq V^\omega$ we define $\mathrm{Pref}(\rho) = \{w \in V^* \mid w \sqsubseteq \rho\}$ and $\mathrm{Pref}(L) = \bigcup_{\rho \in L} \mathrm{Pref}(\rho)$. For $w = w_1 \cdots w_n \in V^+$, let $\mathrm{Last}(w) = w_n$.

[3] This reachability game was the object of McNaughton's study of games playable by humans, which, for practical reasons, should end after a bounded number of steps.