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Interval temporal logics over strongly discrete linear orders: Expressiveness and complexity

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ABSTRACT

Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. Their computational behavior mainly depends on two parameters: the set of modalities they feature and the linear orders over which they are interpreted. In this paper, we identify all fragments of Halpern and Shoham's interval temporal logic HS with a decidable satisfiability problem over the class of strongly discrete linear orders as well as over its relevant subclasses (the class of finite linear orders, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^-$). We classify them in terms of both their relative expressive power and their complexity, which ranges from NP-completeness to non-primitive recursiveness.

1. Introduction

Most temporal logics proposed in the literature assume a point-based model of time. They have been successfully applied in a variety of fields, ranging from the specification and verification of communication protocols to temporal data mining. However, a number of relevant application domains, such as, for instance, those of planning and synthesis of controllers, are often characterized by advanced features like durative actions, and their temporal relationships, accomplishments, and temporal aggregations, which are neglected or dealt with in an unsatisfactory way by point-based formalisms [1]. Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulas relative to time intervals, rather than time points. Interval logic modalities correspond to various relations between pairs of intervals. In particular, the well-known logic HS [2] features a set of modalities that make it possible to express all Allen's interval relations [3]. Interval-based formalisms have been extensively used in various areas of computer science and AI, such as, for instance, specification and verification of reactive systems, temporal databases, theories of action and change, natural language processing, and constraint satisfaction. However, most of them make severe syntactic and/or semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority

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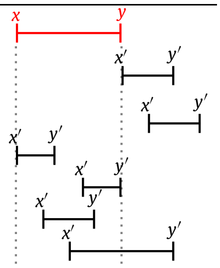
Table 1
Allen's interval relations and corresponding HS modalities.

Relation	Op	Formal definition	Example
meets	$\langle A\rangle$	$[x, y]R_A[x', y'] \Leftrightarrow y = x'$	
before	$\langle L\rangle$	$[x, y]R_L[x', y'] \Leftrightarrow y < x'$	
started-by	$\langle B\rangle$	$[x, y]R_B[x', y'] \Leftrightarrow x = x', y' < y$	
finished-by	$\langle E\rangle$	$[x, y]R_E[x', y'] \Leftrightarrow y = y', x < x'$	
contains	$\langle D\rangle$	$[x, y]R_D[x', y'] \Leftrightarrow x < x', y' < y$	
overlaps	$\langle O\rangle$	$[x, y]R_O[x', y'] \Leftrightarrow x < x' < y < y'$	

of its fragments, turn out to be undecidable. Among the few decidable cases, we mention Propositional Neighborhood Logic (PNL) [4,5] and the logic D of temporal sub-intervals (over dense linear orders) [6].

The computational properties of any HS fragment mainly depend on two parameters: (i) the set of its interval modalities, and (ii) the class of linear orders over which formulas are interpreted. While the first parameter is fairly natural, the second is definitely less obvious. In most cases, the computational behavior of an interval logic does not change when we move from one class of linear orders to another. However, some meaningful exceptions exist. A real character is the logic D: its satisfiability problem is PSPACE-complete over the class of dense linear orders and undecidable over the classes of finite and (weakly) discrete ones (and its status over the class of all linear orders is still unknown). In the last years, the decidability of interval temporal logics has been extensively studied with respect to various meaningful classes of linear orders, including the class of finite linear orders, the class of strongly discrete linear orders (there is a finite number of points between any pair of points), the class of weakly discrete linear orders (every point with a successor/predecessor has an immediate successor/predecessor), which includes non-standard temporal structures like, for instance, $\mathbb{N} + \{\omega\}$, the class of dense linear orders, and the class of all linear orders, plus some temporal structures of special interest like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

In this paper, we focus our attention on the class of strongly discrete linear orders and its relevant subclasses, namely, the class of finite linear orders, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^-$ (the set of all negative integers). Strongly discrete linear orders come into play in a variety of application domains. Consider, for instance, planning problems. They consist of finding a finite partially-ordered sequence of actions that leads the system from the initial to the final state (goal) within a bounded amount of time, satisfying suitable conditions about which sequence of states the world must go through. In this scenario, finite linear orders are usually the most natural option for time modeling. In other fields, such as, for instance, specification and verification of reactive systems, the system is supposed to run forever, starting from some initial state, satisfying a number of safety and response properties. In this case, $\mathbb{N}$ may be the most appropriate choice.

The aim of this paper is twofold: (i) to give a complete picture of HS fragments with respect to decidability/undecidability of their satisfiability problem over the considered cases, filling in the remaining gaps, and (ii) to identify the set of all expressively-different decidable fragments and to determine their exact complexity. In the subsequent sections, we first give a short account of notation and basic notions. Then we review known results, pointing out those HS fragments for which we have incomplete information. Next, we study the expressive power (with respect to modal definability) of all decidable fragments with respect to all classes of linear orders considered in the paper. Sections 5–7 are devoted to decidability/undecidability and complexity results, given in increasing order of complexity (from NP to undecidable). Conclusions provide an assessment of the achieved results.

2. Preliminaries

Let $\mathbb{D} = \langle D, <\rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in D$ and $x < y$ (*strict semantics*).[1] Excluding equality, there are 12 different non-trivial relative position relations between pairs of intervals in a linear order, often called *Allen's relations* [3]: the six relations depicted in Table 1 and the inverse ones. In modal interval temporal logics, interval structures are interpreted as Kripke structures and Allen's relations as accessibility relations, thus associating a modality with each Allen's relation R_X. Formally, for each relation R_X in Table 1, we introduce a modality $\langle X\rangle$ for R_X and a *transposed modality* $\langle\overline{X}\rangle$ for the inverse relation $R_{\overline{X}}$ (that is, $R_{\overline{X}} = (R_X)^{-1}$).

Halpern and Shoham's logic HS is a multi-modal logic with formulas built on a set $\mathcal{AP}$ of proposition letters, the Boolean connectives $\vee$ and $\neg$, and a modality for each Allen's relation. We denote by $\mathsf{X}_1 \ldots \mathsf{X}_k$ the fragment of HS featuring a modality for each Allen's relation in the set $\{R_{X_1}, \ldots, R_{X_k}\}$. Formulas of $\mathsf{X}_1 \ldots \mathsf{X}_k$ are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1\rangle\varphi \mid \ldots \mid \langle X_k\rangle\varphi.$$

[1] Strict semantics excludes intervals with coincident endpoints (point intervals). A *non-strict semantics*, including point intervals, can be possibly adopted. Even though most results can be easily rephrased in this alternative setting, strict semantics is definitely cleaner. Moreover, it is coherent with recent developments in temporal logic that consider points and intervals as different semantic entities (see, e.g., [7]).