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# Complexity of conflict-free colorings of graphs

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ABSTRACT

We consider conflict-free colorings of graph neighborhoods: Each vertex of the graph must be assigned a color so that for each vertex $v$ there is at least one color appearing exactly once in the neighborhood of $v$. The goal is to minimize the number of used colors. We consider both the case of closed neighborhoods, when the neighborhood of a node includes the node itself, and the case of open neighborhoods when a node does not belong to its neighborhood. In this paper, we study complexity aspects of the problem. We show that the problem of conflict-free coloring of closed neighborhoods is NP-complete. Moreover, we give non-approximability results for the conflict-free coloring of open neighborhoods. From a positive point of view, both problems become tractable if parameterized by the vertex cover number or the neighborhood diversity number of the graph. We present simple algorithms which improve on existing results.



## 1. Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with vertex set $V$ and edge set $\mathcal{E}$. A coloring of (the vertices of) $\mathcal{H}$ is a function $C : V \to \mathbb{Z}^+$. A $k$-coloring of $\mathcal{H}$ is a coloring $C$ with $|C(V)| \le k$, for some $k \ge 1$. A coloring is called proper if no edge $e \in \mathcal{E}$ containing at least two vertices is monochromatic. The smallest number of colors needed to properly color the vertices of $\mathcal{H}$ is called the chromatic number of $\mathcal{H}$ and is denoted by $\chi(\mathcal{H})$. A coloring is said to be conflict-free if every hyperedge contains a vertex whose color is unique among those assigned to the vertices of the hyperedge.

**Definition 1** *(CF coloring).* Let $C$ be a coloring of a hypergraph $\mathcal{H} = (V, \mathcal{E})$. $C$ is a *conflict-free coloring* of $\mathcal{H}$ if for each $e \in \mathcal{E}$ there exists a vertex $v \in e$ such that $C(u) \ne C(v)$ for any $u \in e$ with $u \ne v$.

The study of conflict-free colorings was initially motivated by a frequency assignment problem in cellular networks [10]. Such networks consist of fixed-position *base stations*, each assigned a fixed frequency, and roaming *clients*. Roaming clients have a range of communication and come under the influence of different subsets of base stations. This situation can be modeled by means of a hypergraph whose vertices correspond to the base stations. The range of communication of a mobile agent, that is, the set of base stations it can communicate with, is represented by a hyperedge $e \in \mathcal{E}$. A CF-coloring of such a hypergraph implies an assignment of frequencies, to the base stations, which enables clients to connect to a base station holding a unique frequency in the client's range, thus avoiding interferences. The goal is to minimize the number of assigned frequencies.

CF-coloring also finds application in RFID (Radio Frequency Identification) networks. RFID allows a reader device to sense the presence of a nearby object by reading a tag attached to the object itself. To improve coverage, multiple RFID readers

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can be deployed in an area. However, two readers trying to access a tagged device simultaneously might cause mutual interference. It can be shown that CF-coloring of the readers can be used to assure that every possible tag will have a time slot and a single reader trying to access it in that time slot [23].

Due to both its practical motivations and its theoretical interest, conflict-free coloring has been the subject of several papers; a survey of results in the area is given in [23]. The theoretic study of the CF-chromatic number in general graphs and hypergraphs was initiated in [22] and has recently raised much interest due to the novel combinatorial and algorithmic questions it poses, see [2–4,6,15,16,18].

### 1.1. CF-colorings of neighborhoods

In this paper we study the conflict-free coloring of hypergraphs induced by the neighborhoods of the vertices of a graph [22].

Given a graph $G=(V,E)$ and a vertex $u \in V$, the *open neighborhood* $N_G(u)$ of $u$ is defined as the set consisting of all the vertices in $G$ connected to $u$. The set $N_G[u]=N_G(u)\cup\{u\}$ is called the *closed neighborhood* of $u$. We will write $N(u)$ and $N[u]$ whenever $G$ is clear from the context.

A conflict-free coloring with respect to the open (resp. closed) neighborhoods of $G$ is defined as the conflict-free coloring of the hypergraph with vertex set $V$ and edge set $\{N_G(u) \mid u \in V\}$ (resp. $\{N_G[u] \mid u \in V\}$). For the sake of simplicity, we now reformulate Definition 1 in terms of the graph $G$.

Given a graph $G=(V,E)$ and a coloring $C$, we say that the set $U \subseteq V$ has a *unique color* under $C$ if there exists a color $c$ such that $|\{v \in U \mid C(v)=c\}|=1$. Equivalently, we say that $c$ is unique for $U$. All the graphs considered in this paper are supposed to be connected.

**Definition 2.** Consider a graph $G=(V,E)$.

**CF-ON** coloring: A coloring $C$ is called conflict-free with respect to the *open neighborhoods* of $G$ if for each $u \in V$ the set $N(u)$ has a *unique color* under $C$.

**CF-CN** coloring: A coloring $C$ is called conflict-free with respect to the *closed neighborhoods* of $G$ if for each $u \in V$ the set $N[u]$ has a *unique color* under $C$.

The smallest number of colors needed by any possible CF-ON (resp. CF-CN) coloring of $G$ is called the CF-ON (resp. CF-CN) chromatic number of $G$ and is denoted by $\chi_{\mathrm{CF}}(G)$ (resp. $\chi_{\mathrm{CF}}[G]$).

It is possible to show (see also [22]) that, given a graph $G$, the same greedy upper bound $\Delta_G+1$ (where $\Delta_G$ is the maximum degree of a vertex in $G$) holds for the chromatic number, the CF-ON chromatic number, and the CF-CN chromatic number. However, as also noticed in [22], these values can be quite different and no ordering among them is valid for any graph. Examples are given in Fig. 1. Consider first the complete graph $K_n$ on $n$ vertices, it is not difficult to see that

$$\chi_{\mathrm{CF}}[K_n]=2<\chi_{\mathrm{CF}}(K_n)=3<\chi(K_n)=n.$$

Moreover, it is easy to show that for any tree $T$

$$\chi_{\mathrm{CF}}[T]=\chi_{\mathrm{CF}}(T)=\chi(T)=2.$$

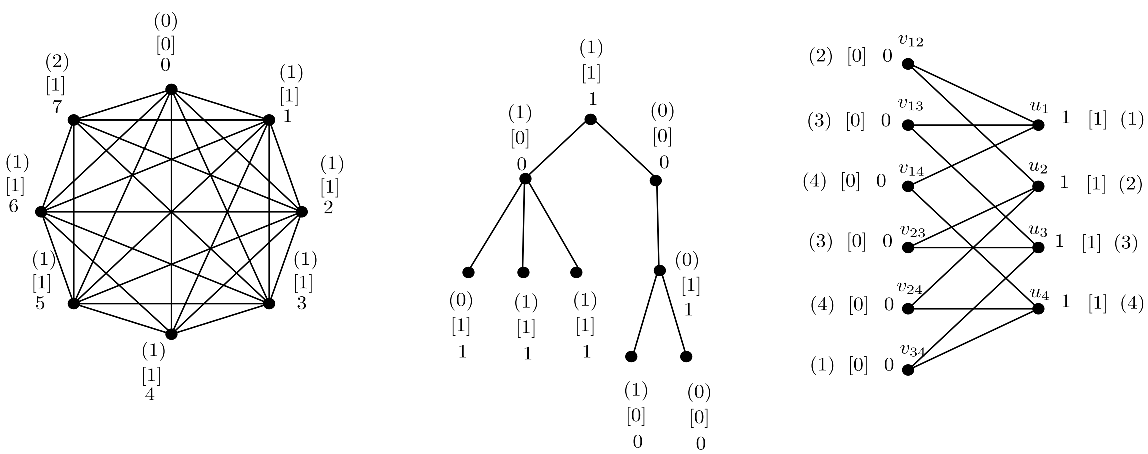


**Fig. 1.** The graph $K_8$, a tree $T$, and the graph $B_4$ with the corresponding colorings. For each node $c, [c'], (c'')$ represent the colors assigned to the node in a proper, a CF-CN, and a CF-ON coloring, respectively.