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Generalized rainbow connectivity of graphs ☆

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ABSTRACT

Let $C = \{c_1, c_2, \dots, c_k\}$ be a set of k colors, and let $\vec{\ell} = (\ell_1, \ell_2, \dots, \ell_k)$ be a k-tuple of nonnegative integers $\ell_1, \ell_2, \dots, \ell_k$. For a graph $G = (V, E)$, let $f : E \to C$ be an edge-coloring of G in which two adjacent edges may have the same color. Then, the graph G edge-colored by f is $\vec{\ell}$-rainbow connected if every two vertices of G have a path P connecting them such that the number of edges on P that are colored with c_j is at most ℓ_j for each index $j \in \{1, 2, \dots, k\}$. Given a k-tuple $\vec{\ell}$ and an edge-colored graph, we study the problem of determining whether the edge-colored graph is $\vec{\ell}$-rainbow connected. In this paper, we first study the computational complexity of the problem with regard to certain graph classes: the problem is NP-complete even for cacti, while is solvable in polynomial time for trees. We then give an FPT algorithm for general graphs when parameterized by both k and $\ell_{\max} = \max\{\ell_j \mid 1 \leqslant j \leqslant k\}$.

1. Introduction

Graph connectivity is one of the most fundamental graph-theoretic properties. In the literature, several measures for graph connectivity have been studied, such as requiring hamiltonicity, edge-disjoint spanning trees, or edge- or vertex-cuts of sufficiently large size. Recently, there has been some interest in studying problems on colored graphs, due to their applications in areas such as computational biology, transportation and telecommunications [9]. In this paper, we generalize an interesting concept of graph connectivity on colored graphs, called the *rainbow connectivity*, which was introduced by Chartrand et al. [6] and has been extensively studied in the literature [2,4–8,11,12].

Let $G = (V, E)$ be a graph with vertex set V and edge set E; we often denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. Let $C = \{c_1, c_2, \dots, c_k\}$ be a set of k colors, and let $\vec{\ell} = (\ell_1, \ell_2, \dots, \ell_k)$ be a k-tuple of nonnegative integers $\ell_1, \ell_2, \dots, \ell_k$. Consider a mapping $f : E \to C$, called an *edge-coloring* of G. Note that f is not necessarily a proper edge-coloring, that is, f may assign a same color to two adjacent edges. We denote by $G(f)$ the graph G edge-colored by f. Then, a path P in $G(f)$ connecting two vertices u and v in V is called an *$\vec{\ell}$-rainbow path* between u and v if the number of edges in P that are colored with c_j is at most ℓ_j for every index $j \in \{1, 2, \dots, k\}$. The edge-colored graph $G(f)$ is *$\vec{\ell}$-rainbow connected* if $G(f)$ has an $\vec{\ell}$-rainbow path between every two vertices in V. Note that these $\vec{\ell}$-rainbow paths are not necessarily edge-disjoint for pairs of vertices. For example, the edge-colored graph $G(f)$ in Fig. 1 is $\vec{\ell}$-rainbow connected for $\vec{\ell} = (1, 3, 2)$.

The concept of $\vec{\ell}$-rainbow connectivity was originally introduced by Chartrand et al. [6] for the special case where $\vec{\ell} = (1, 1, \dots, 1)$. Chakraborty et al. [4] defined the RAINBOW CONNECTIVITY problem which asks whether a given edge-colored

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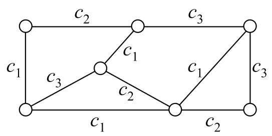

Fig. 1. An $\vec{\ell}$-rainbow connected graph, where $\vec{\ell} = (1, 3, 2)$.

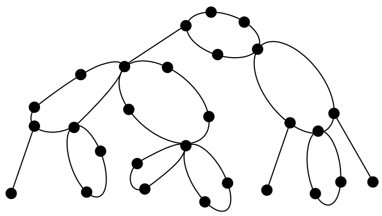

Fig. 2. An example of cacti.

graph is $(1, 1, \ldots, 1)$-rainbow connected, and showed that the problem is NP-complete in general. Then, Uchizawa et al. [12] studied the computational complexity of the problem with regard to certain graph classes, and also settled it with regard to graph diameters. (Remember that the *diameter* of a graph G is the maximum number of edges in a shortest path between any two vertices in G.)

In this paper, we introduce and study the following generalized problem: Given a k-tuple $\vec{\ell}$ and an edge-coloring f of a graph G, the GENERALIZED RAINBOW CONNECTIVITY problem is to determine whether $G(f)$ is $\vec{\ell}$-rainbow connected. Thus, (ordinary) RAINBOW CONNECTIVITY is a specialization of GENERALIZED RAINBOW CONNECTIVITY. We first give precise complexity analyses for GENERALIZED RAINBOW CONNECTIVITY with regard to certain graph classes. We then give an FPT algorithm for the problem on general graphs when parameterized by both $k = |C|$ and $\ell_{\max} = \max\{\ell_j \mid 1 \leqslant j \leqslant k\}$. (An early version of the paper was presented in [13].) Below we explain our results more precisely, together with comparisons with known results [12].

Graph classes

From the viewpoint of graph classes, we clarify a boundary on graph classes between tractability and NP-completeness: GENERALIZED RAINBOW CONNECTIVITY is NP-complete even for cacti, while there is a polynomial-time algorithm for trees. Note that trees and cacti are very close to each other in the following sense: trees form a graph class which is a subclass of cacti, and the "treewidth" of cacti is two [3]. It is remarkable that the boundary is different from the known one for RAINBOW CONNECTIVITY [12]: it is NP-complete for outerplanar graphs, and is solvable in polynomial time for cacti. Therefore, the NP-complete proof given by [12] does not imply our result. We also remark that our polynomial-time algorithm for trees is always faster than a naive one, as discussed in Section 3.1.

FPT algorithm

In Section 3.2, we give an algorithm which solves GENERALIZED RAINBOW CONNECTIVITY for general graphs in time $O(k2^{k\ell_{\max}}mn)$ using $O(kn2^{k\ell_{\max}}\log(\ell_{\max}+1))$ space, where n and m are the numbers of vertices and edges in a graph, respectively. Therefore, the problem can be solved in polynomial time for the following two cases: (a) $k = O(\log n)$ and $\ell_{\max}$ is a fixed constant; and (b) k is a fixed constant and $\ell_{\max} = O(\log n)$. We remark that our FPT algorithm generalizes the known one [12]: the same running time and space complexity of the known FPT algorithm for RAINBOW CONNECTIVITY [12] can be obtained from our result as the special case where $\ell_{\max} = 1$.

2. NP-completeness for cacti

A graph G is a *cactus* if every edge is part of at most one cycle in G [3]. (See Fig. 2 as an example of cacti.) The main result of this section is the following theorem.

Theorem 1. GENERALIZED RAINBOW CONNECTIVITY *is* NP-*complete even for cacti and* $\vec{\ell} = (2, 2, \ldots, 2)$.

In the remainder of this section, we give a proof of Theorem 1.

Let $G(f)$ be a given edge-colored graph. We can clearly check in polynomial time whether a given path in $G(f)$ is an $\vec{\ell}$-rainbow path, and hence GENERALIZED RAINBOW CONNECTIVITY belongs to NP. We below show that the problem is NP-hard even for cacti and $\vec{\ell} = (2, 2, \ldots, 2)$ by a polynomial-time reduction from the 3-OCCURRENCE 3SAT problem defined as follows: Given a 3CNF formula ϕ such that each variable appears at most three times in ϕ, determine whether ϕ is satisfiable. 3-OCCURRENCE 3SAT is known to be NP-complete [10].