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Fixed-orientation equilateral triangle matching of point sets

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ABSTRACT

Given a point set P and a class $\mathscr{C}$ of geometric objects, $G_{\mathscr{C}}(P)$ is a geometric graph with vertex set P such that any two vertices p and q are adjacent if and only if there is some $C \in \mathscr{C}$ containing both p and q but no other points from P. We study $G_{\triangledown}(P)$ graphs where $\triangledown$ is the class of downward equilateral triangles (i.e., equilateral triangles with one of their sides parallel to the x-axis and the corner opposite to this side below that side). For point sets in general position, these graphs have been shown to be equivalent to half-Θ_6 graphs and TD-Delaunay graphs.

The main result in our paper is that for point sets P in general position, $G_{\triangledown}(P)$ always contains a matching of size at least $\lceil \frac{|P|-1}{3} \rceil$ and this bound is tight. We also give some structural properties of $G_{✡}(P)$ graphs, where ✡ is the class which contains both upward and downward equilateral triangles. We show that for point sets in general position, the block cut point graph of $G_{✡}(P)$ is simply a path. Through the equivalence of $G_{✡}(P)$ graphs with Θ_6 graphs, we also derive that any Θ_6 graph can have at most $5n-11$ edges, for point sets in general position.

1. Introduction

In this work, we study the structural properties of some special geometric graphs defined on a set P of n points on the plane. An equilateral triangle with one side parallel to the x-axis and the corner opposite to this side below (resp. above) that side as in $\triangledown$ (resp. $\triangle$) will be called a down (resp. up)-triangle. A point set P is said to be in general position, if the line passing through any two points from P does not make angles $0°$, $60°$ or $120°$ with the horizontal [1,2]. In this paper, we consider only point sets that are in general position and our results assume this pre-condition.

Given a point set P, $G_{\triangledown}(P)$ (resp. $G_{\triangle}(P)$) is defined as the graph whose vertex set is P and that has an edge between any two vertices p and q if and only if there is a down-(resp. up-)triangle containing both points p and q but no other points from P (see Fig. 1). We also define another graph $G_{✡}(P)$ as the graph whose vertex set is P and that has an edge between any two vertices p and q if and only if there is a down-triangle or an up-triangle containing both points p and q but no other points from P. In Section 3 we will see that, for any point set P in general position, its $G_{\triangledown}(P)$ graph is the same as the well known Triangle Distance Delaunay (TD-Delaunay) graph of P and the half-Θ_6 graph of P on so-called negative cones. Moreover, $G_{✡}(P)$ is the same as the Θ_6 graph of P [1,3].

Given a point set P and a class $\mathscr{C}$ of geometric objects, the maximum $\mathscr{C}$-matching problem is to compute a subclass $\mathscr{C}'$ of $\mathscr{C}$ of maximum cardinality such that no point from P belongs to more than one element of $\mathscr{C}'$ and for each $C \in \mathscr{C}'$, there are exactly two points from P which lie inside C. Dillencourt [4] proved that every point set admits a perfect

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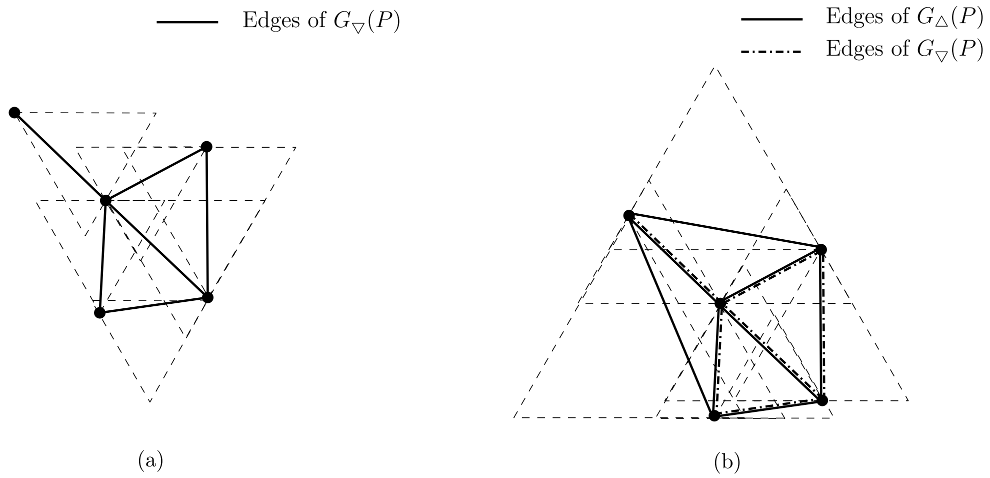

Fig. 1. A point set P and its (a) $G_{\bigtriangledown}(P)$ and (b) $G_{\davidsstar}(P)$.

circle-matching. Ábrego et al. [5] studied the isothetic square matching problem. Bereg et al. concentrated on matching points using axis-aligned squares and rectangles [6].

A matching in a graph G is a subset M of the edge set of G such that no two edges in M share a common end-point. A matching is called a maximum matching if its cardinality is the maximum among all possible matchings in G. If all vertices of G appear as end-points of some edge in the matching, then it is called a perfect matching. It is not difficult to see that for a class $\mathscr{C}$ of geometric objects, computing the maximum $\mathscr{C}$-matching of a point set P is equivalent to computing the maximum matching in the graph $G_{\mathscr{C}}(P)$.

The maximum $\bigtriangleup$-matching problem, which is the same as the maximum matching problem on $G_{\bigtriangleup}(P)$, was previously studied by Panahi et al. [2]. It was claimed that, for any point set P of n points in general position, any maximum matching of $G_{\bigtriangleup}(P)$ (and $G_{\bigtriangledown}(P)$) will match at least $\lfloor \frac{2n}{3} \rfloor$ vertices. But we found that their proof of Lemma 7, which is very crucial for their result, has gaps. By a completely different approach, we show that for any point set P in general position, $G_{\bigtriangledown}(P)$ (and by symmetric arguments, $G_{\bigtriangleup}(P)$) will have a maximum matching of size at least $\lceil \frac{n-1}{3} \rceil$; i.e., at least $2(\lceil \frac{n-1}{3} \rceil)$ vertices are matched. We also give examples of point sets, where our bound is tight.

We also prove some structural and geometric properties of the graphs $G_{\bigtriangledown}(P)$ (and by symmetric arguments, $G_{\bigtriangleup}(P)$) and $G_{\davidsstar}(P)$. It will follow that for point sets in general position, Θ_6 graphs can have at most $5n-11$ edges and their block cut point graph is a simple path.

2. Notations

Our notations are similar to those used in [1], with some minor modifications adopted for convenience. A *cone* is the region in the plane between two rays that emanate from the same point, its apex. Consider the rays obtained by a counter-clockwise rotation of the positive x-axis by angles of $\frac{i\pi}{3}$ with $i = 1, \ldots, 6$ around a point p (see Fig. 2). Each pair of successive rays, $\frac{(i-1)\pi}{3}$ and $\frac{i\pi}{3}$, defines a cone, denoted by $A_i(p)$, whose apex is p. For $i \in \{1, \ldots, 6\}$, when i is odd, we denote $A_i(p)$ using $C_{\frac{i+1}{2}}(p)$ and the cone opposite to $C_i(p)$ using $\overline{C}_i(p)$. We call $C_i(p)$ a positive cone around p and $\overline{C_i}(p)$ a negative cone around p. For each cone $\overline{C_i}(p)$ (resp. $C_i(p)$), let $\ell_{\overline{C_i}(p)}$ (resp. $\ell_{C_i(p)}$) be its bisector. If $p' \in \overline{C_i}(p)$, then let $\overline{c_i}(p, p')$ denote the distance between p and the orthogonal projection of p' onto $\ell_{\overline{C_i}(p)}$. Similarly, if $p' \in C_i(p)$, then let $c_i(p, p')$ denote the distance between p and the orthogonal projection of p' onto $\ell_{C_i(p)}$. For $1 \leqslant i \leqslant 3$, let $V_i(p) = \{p' \in P \mid p' \in C_i(p),\ \ p' \neq p\}$ and $\overline{V_i}(p) = \{p' \in P \mid p' \in \overline{C_i}(p),\ \ p' \neq p\}$. For any two points p and q, the smallest down-triangle containing p and q is denoted by $\bigtriangledown pq$ and the smallest up-triangle containing p and q is denoted by $\bigtriangleup pq$. If G_1 and G_2 are graphs on the same vertex set, $G_1 \cap G_2$ (resp. $G_1 \cup G_2$) denotes the graph on the same vertex set whose edge set is the intersection (resp. union) of the edge sets of G_1 and G_2.

3. Preliminaries

In this section, we describe some basic properties of the geometric graphs described earlier and their equivalence with other geometric graphs which are well known in the literature.

The class of down-triangles (and up-triangles) admits a shrinkability property [5]: each triangle object in this class that contains two points p and q, can be shrunk such that p and q lie on its boundary. It is also clear that we can continue the shrinking process—from the edge that does not contain neither p or q—until at least one of the points, p or q, becomes a triangle vertex and the other point lies on the edge opposite to this vertex. After this, if we shrink the triangle further,