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On the number of minimal dominating sets on some graph classes

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A B S T R A C T

A dominating set in a graph is a subset of vertices such that each vertex is either in the dominating set or adjacent to some vertex in the dominating set. It is known that graphs have at most $O(1.7159^n)$ minimal dominating sets. In this paper, we establish upper bounds on this maximum number of minimal dominating sets for split graphs, cobipartite graphs and interval graphs. For each of these graph classes, we provide an algorithm to enumerate them. For split and interval graphs, we show that the number of minimal dominating sets is at most $3^{n/3} \approx 1.4423^n$, which is the best possible bound. This settles a conjecture by Couturier et al. (SOFSEM 2012, [1]). For cobipartite graphs, we lower the $O(1.5875^n)$ upper bound from Couturier et al. to $O(1.4511^n)$.

1. Introduction

The enumeration of combinatorial objects in graphs dates back to the sixties and found a growing interest in the literature. Among these objects, maximal independent sets, minimal dominating sets, minimal separators, potential maximal cliques, minimal feedback vertex sets, minimal transversals in hypergraphs have been studied [2–6]. A major question deals with the maximum number of such objects a graph can have. A famous result, due to Moon and Moser [2] in 1965, establishes that the maximum number of maximal independent sets in an n-vertex graph is $3^{n/3}$.

In this paper, we focus on minimal dominating sets. Given a graph $G = (V, E)$, a subset $D \subseteq V$ is a dominating set if each vertex is either in D or has at least one neighbor in D. The set D is minimal if no proper subset $D' \subset D$ is a dominating set. We establish bounds on the maximum number of minimal dominating sets on various graph classes: split graphs, interval graphs and cobipartite graphs. The enumeration of dominating sets attracts interest in the literature and is useful for solving other problems, like *e.g.* computing the domatic number of a graph [7]. In [7] it is shown that a graph has at most $O(1.7159^n)$ minimal dominating sets and there exist graphs with at least $O(1.5705^n)$ minimal dominating sets. Recently, the enumeration of minimal dominating sets has been studied for several graph classes. A summary of up-to-date results (from [1,8,7] and [9]) is given in Table 1. We also give in this table our new results proven in this paper. To establish these new bounds, we provide exponential-time algorithms to enumerate all the minimal dominating sets in the considered graph classes, and their worst-case running-time analyses. Finally, let us mention that the enumeration of minimal dominating sets has also been considered under the scope of polynomial delay and output-polynomial time algorithms (see [10–12]).

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Table 1
Lower and upper bounds on the maximum number of minimal dominating sets. We note that the lower bound of $15^{n/6}$ for general graphs is stated in [7] and due to Dieter Kratsch. Note that $3^{n/3} \approx 1.4423^n$ and $15^{n/6} \approx 1.5704^n$.

Graph class	Lower bound	Ref.	Upper bound	Ref.
general	$15^{n/6}$	[7]	1.7159^n	[7]
tree	1.4167^n	[9]	1.4656^n	[9]
chordal	$3^{n/3}$	[8]	1.6181^n	
cobipartite	1.3195^n	[1]	1.5875^n	[1]
			$n^2 + 2 \cdot 1.4511^n$	[this paper]
split	$3^{n/3}$	[8]	1.4656^n	[8]
			$3^{n/3}$	[this paper]
proper interval	$3^{n/3}$	[8]	1.4656^n	[8]
interval	$3^{n/3}$	[8]	$3^{n/3}$	[this paper]
cograph	$15^{n/6}$	[8]	$15^{n/6}$	[8]
trivially perfect	$3^{n/3}$	[8]	$3^{n/3}$	[8]
threshold	$\omega(G)$	[8]	$\omega(G)$	[8]
chain	$\lfloor n/2 \rfloor + m$	[8]	$\lfloor n/2 \rfloor + m$	[8]

In this paper, we settle the following conjecture stated by Couturier et al. in [8]:

Conjecture. *(See [8].) The maximum number of minimal dominating sets in proper interval graphs and in split graphs is* $3^{n/3}$.

More precisely, we prove that the number of minimal dominating sets in split graphs and even in (not necessary proper) interval graphs is at most $3^{n/3}$. Notice that proper interval graphs is a subclass of interval graphs. We also improve on the upper-bound from [1] on the maximum number of minimal dominating sets for cobipartite graphs, from 1.5875^n to $n^2 + 2 \cdot 1.4511^n$.

2. Preliminaries

Let $G = (V, E)$ be a graph, $n = |V|$ and $m = |E|$. Given a vertex v, we denote by $N(v)$ the set of its neighbors and by $N[v] = N(v) \cup \{v\}$ its closed neighborhood. For a subset $D \subseteq V$, the set $N[D]$ is defined as $\cup_{v \in D} N[v]$. A set D is called a *dominating set* if $N[D] = V$. Given a vertex v and a set X, we set $N_X(v) = N(v) \cap X$ and $d_X(v) = |N_X(v)|$. For a set D, a vertex $x \in D$ and a vertex $v \in V$ (where eventually, $x = v$), we say that v is a *private neighbor* of x if $v \notin N[D \setminus \{x\}]$.

We observe that if a dominating set D contains a vertex $x \in D$ with no private neighbor, then the set $D \setminus \{x\}$ is also a dominating set. Conversely, it is easy to observe that if each vertex of a dominating set D has a private neighbor, then D is a minimal dominating set. This suggests the following property which will be useful throughout the paper:

Property *($\star$). A dominating set D is minimal if and only if each vertex of the dominating set D has a private neighbor.*

In the forthcoming sections, we consider some graph classes. Here we only recall their definitions; for more definitions or properties, we refer to [13,14].

A subset $S \subseteq V$ is a clique (an independent set) if the vertices of S are pairwise adjacent (non-adjacent, respectively). A graph G is a split graph (a cobipartite graph) if its vertex set V can be partitioned into two sets X and Y, where X is a clique and Y an independent set (X and Y are cliques, respectively). A graph G is an interval graph if an interval of the real line can be assigned to each vertex of G, such that two vertices are adjacent if and only if their corresponding intervals intersect. In such a graph, for each vertex v, we note by $l(v)$ (or $r(v)$) the value of the left endpoint (respectively, of the right endpoint) of the interval of the real line corresponding to v. An interval model of G is normalized if $\bigcup_{v \in V} \{l(v), r(v)\} = \{1, 2, \ldots, 2n\}$.

Two graphs $G = (V_G, E_H)$ and $H = (V_H, E_H)$ are isomorphic, denoted by $G \cong H$, if there exists a bijection f between V_G and V_H such that $\{u, v\} \in E_G$ if and only if $\{f(u), f(v)\} \in E_H$.

Branching algorithms To enumerate minimal dominating sets of a given graph G, we design branching algorithms. The execution of such an algorithm can be viewed as a search tree, whose leaves correspond to the minimal dominating sets of the input graph. Typically, such algorithms consist in a collection of branching rules, with recursive calls on some sub-instances. By upper bounding the maximum number of leaves in this search tree, we also establish an upper bound on the maximum number of minimal dominating sets. Such bound on the number of leaves is obtained via linear recurrences. To analyze the running-time of our branching algorithms it is sufficient to observe that in the search tree, each path from a leaf to the root (which corresponds to the input graph) is of polynomial length. Moreover, the time spent by the algorithm on a subproblem (which corresponds to a node of the search tree) is polynomial, excluding the time consumed by the recursive calls. Thus the running-time is at most a polynomial times the number of leaves, *i.e.* the maximum number of minimal dominating sets up to a polynomial. The analysis of our branching algorithms is done via linear recurrences; we provide basic definitions