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Note

Remarks concerning the freeness problem over morphism and matrix semigroups

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ABSTRACT

We study the freeness problem over morphism and matrix semigroups. We show that the freeness problem is undecidable for morphisms over a three-letter alphabet. We show that there is a commutative semiring R such that the freeness problem is undecidable for upper-triangular 2×2 matrices having entries in R.

1. Introduction

In this paper we study the freeness problem over semigroups. In general, the freeness problem over a semigroup S with a recursive underlying set can be stated as follows: given a finite alphabet Σ and a morphism $\sigma : \Sigma^+ \to S$ decide whether or not σ is injective.

For a general introduction to this topic see [3].

We will study the freeness problem for semigroups consisting of morphisms and for matrix semigroups. We show that the freeness problem is undecidable for semigroups consisting of morphisms over a three-letter alphabet. This is related to Open question 1 in [3].

We will also discuss the freeness problem for upper-triangular 2×2 matrices. In [3] it is asked whether there exists a commutative semiring R (satisfying some additional conditions) such that the freeness problem is undecidable for 2×2 matrices having entries in R. We will show that a simple rewriting of the undecidability proof given in [2] shows that actually the simple semiring $\mathbb{N} \times \mathbb{N}$ consisting of tuples of nonnegative integers has the desired properties.

2. Definitions and earlier results

We use standard language-theoretic notation and terminology. In particular, the length of a word w is denoted by $|w|$ and the empty word is denoted by ε.

Let X be an alphabet and let $L \subseteq X^*$. Then $\mathrm{Alph}(L)$ is the smallest subset Y of X such that $L \subseteq Y^*$.

As usual, $\mathbb{N}$ is the set of nonnegative integers. Suppose $n \geq 2$ is an integer. If s is a nonnegative integer and $a_0, \ldots, a_{s-1} \in \mathbb{N}$, then the value of the word $w = a_0 a_1 \ldots a_{s-1}$ in base n is the number

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$$\mathrm{val}_n(w) = a_0 + a_1 n + \cdots + a_{s-1} n^{s-1}.$$

Observe that when using the notation val_n we do not assume that the digits are smaller than n. Instead, we allow the digits to be arbitrary nonnegative integers. It is clear, for example, that val_2 gives an injective mapping from $\{1,2\}^*$ into $\mathbb{N}$.

For any alphabet X, let $\mathrm{Hom}(X^*)$ be the set of all morphisms from X^* to itself. $\mathrm{Hom}(X^*)$ is a monoid with respect to the usual product of morphisms.

If R is a semiring and n is a positive integer, $R^{n\times n}$ is the set of $n \times n$ matrices having entries in R and $\mathrm{Tri}(n, R)$ is the subset of $R^{n\times n}$ consisting of upper-triangular matrices. Further, $\mathrm{Tri}_1(2, R)$ is the subset of $\mathrm{Tri}(2, R)$ consisting of matrices having the identity element of R in the lower right corner.

Following [3] we denote $\mathbb{W} = \{1,2\}^*$. Hence the semigroup $\mathbb{W} \times \mathbb{W}$ is the set of all pairs (u, v) where u and v are words over the binary alphabet $\{1,2\}$. (In [3] the more natural alphabet $\{0,1\}$ is used. To avoid problems with leading zeros we have replaced $\{0,1\}$ by $\{1,2\}$. This change is not needed when considering matrix monoids but it will be useful when we consider monoids consisting of morphisms.)

In what follows S will always be a monoid with a recursive underlying set.

Suppose k is a positive integer and L is a language. We will consider the problems $\mathrm{Free}[S]$, $\mathrm{Free}(k)[S]$ and $\mathrm{Free}[S, L]$ defined as follows.

$\mathrm{Free}[S]$ is the problem: given a finite alphabet Σ and a morphism $\sigma : \Sigma^* \to S$, decide whether or not σ is injective. $\mathrm{Free}(k)[S]$ is obtained from $\mathrm{Free}[S]$ by considering only alphabets having k letters. Hence $\mathrm{Free}(k)[S]$ is the problem: given an alphabet Σ having k letters and a morphism $\sigma : \Sigma^* \to S$ decide whether or not σ is injective. $\mathrm{Free}[S, L]$ is obtained from $\mathrm{Free}[S]$ by considering the injectivity only on the language L. Hence $\mathrm{Free}[S, L]$ is the problem: given a morphism $\sigma : \Sigma^* \to S$ such that $L \subseteq \Sigma^*$ decide whether or not σ is injective on L.

Note that it does not make sense to consider the problems $\mathrm{Free}(k)[S, L]$ for various values of k since the cardinality of $\mathrm{Alph}(L)$ already determines the unique value of k for which the problem is of interest.

More general problems are obtained by considering various classes of languages. If $\mathcal{L}$ is an arbitrary class of languages and k is a positive integer, the problems $\mathrm{Free}[S, \mathcal{L}]$ and $\mathrm{Free}(k)[S, \mathcal{L}]$ are defined in the natural way. The problem $\mathrm{Free}(k)[S, \mathcal{L}]$, for example, is: given a language L in $\mathcal{L}$ such that the cardinality of $\mathrm{Alph}(L)$ is at most k and a morphism $\sigma : \mathrm{Alph}(L)^* \to S$ decide whether or not σ is injective on L.

For an excellent introduction to problems $\mathrm{Free}[S]$ and $\mathrm{Free}(k)[S]$ we refer to [3]. The next three theorems give some important undecidability results concerning the freeness problem. For their proofs see [1–4].

Theorem 1. *For every integer $k \geq 13$, $\mathrm{Free}(k)[\mathrm{Tri}(3, \mathbb{N})]$ is undecidable.*

Theorem 2. *Let $\mathcal{H}$ be the skew field of rational quaternions. Then $\mathrm{Free}(7)[\mathcal{H}^{2\times 2}]$ is undecidable.*

Theorem 3. *For every integer $k \geq 13$, $\mathrm{Free}(k)[\mathbb{W} \times \mathbb{W}]$ is undecidable.*

The decidability status of $\mathrm{Free}[\mathrm{Tri}(2, \mathbb{N})]$ is open (see [2]).

In the next section we will use the following theorem due to [4] which shows that the mixed modification of PCP is undecidable for alphabets having at least seven letters.

Theorem 4. *Let Σ be an alphabet having at least seven letters. It is undecidable for morphisms $h, g : \Sigma^* \to \mathbb{W}$ whether or not there exists a nonempty word $w = a_1 a_2 \ldots a_k$ where $a_i \in \Sigma$ for $1 \leq i \leq k$ such that*

$$h_1(a_1)h_2(a_2)\ldots h_k(a_k) = g_1(a_1)g_2(a_2)\ldots g_k(a_k)$$

where $h_i, g_i \in \{h, g\}$ for $1 \leq i \leq k$ and $h_j \neq g_j$ for at least one index j.

It is sometimes possible to prove results for morphism monoids by using results concerning matrix monoids or vice versa. An example of this is provided by Theorem 2.14 in [3] stating that the morphism torsion problem is decidable. The proof is done by reducing the problem to the matrix torsion problem. The idea can easily be extended to prove that it is decidable whether or not a finitely generated submonoid of a morphism monoid is finite.

3. The freeness problem for morphism monoids

In this section we prove that the freeness problem is undecidable for semigroups consisting of morphisms over a three-letter alphabet.

Theorem 5. *If Δ has at least three letters, then $\mathrm{Free}(14)[\mathrm{Hom}(\Delta^*)]$ is undecidable.*

Proof. Let $\Delta = \{a, b, c\}$ and let $\Sigma = \{x_1, \ldots, x_7\}$. Let $g, h : \Sigma^* \to \mathbb{W}$ be morphisms. Define the morphisms $g_i, h_i : \Delta^* \to \Delta^*$ for $1 \leq i \leq 7$ by