Theoretical Computer Science 539 (2014) 28–37

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Locating a backtracking robber on a tree

Suzanne Seager *

Mount Saint Vincent University, 166 Bedford Highway, Halifax, NS B3M 2J6, Canada

ARTICLE INFO

Article history:
Received 30 September 2013
Received in revised form 10 March 2014
Accepted 14 April 2014
Communicated by D. Peleg

Keywords:
Graph location
Cops and robbers
Graph searching
Metric dimension

ABSTRACT

Carraher, Choi, Delcourt, Erickson, and West (2012) [4] introduced the following version of a robber locating game: At each turn, the cop chooses a vertex of the graph to probe, and receives the distance from the probe to the robber. The cop wins if she can uniquely locate the robber after this probe. Otherwise the robber may stay put or move to any vertex adjacent to his location. We answer some of their conjectures and characterize the trees for which the cop wins.

1. Introduction

In the game of cop and robber, introduced by Nowakowski and Winkler [6] and Quilliot [7], the cop and then the robber choose starting vertices on a connected graph, then take turns playing, at each turn either moving to an adjacent vertex or staying put. The cop's goal is to capture the robber by moving to the robber's vertex and the robber's goal is to avoid being captured. The cop and the robber each know where the other is throughout the game. Many variants of the cop and robber game have been proposed; see Bonato and Nowakowski [2] for a summary.

Graph location, introduced by Slater [10] and Harary and Melter [5], can also be described as a game, in which the first player chooses a vertex and the second player chooses a set of vertices as probes, receiving the distance from each probe to the first player's location. From these distances the second player must locate the first player, the goal being to do so with the minimum number of probes. The original problem required all probes to be chosen in advance, with the minimum number called the *metric dimension* of the graph. A good survey is Cáceres et al. [3], who note that there is also a sequential version of this problem in which the second player chooses one probe at a time, receiving the distance from that probe before choosing the next. We have also studied this version in [8].

Since both cop and robber and graph location have as their goal finding a specific vertex, it was natural to synthesize them into one, the robber locating game, which we introduced in [9]. At the start of this game, the robber chooses a vertex. The cop then chooses a *probe* vertex, and receives the distance from the probe to the robber. If the cop can locate the robber after this probe, she wins. Otherwise, the robber may move to an adjacent vertex or stay put, but may not move to the probe vertex. The cop then chooses a second probe, and so on. The cop wins if she can locate the robber after a finite number of probes; otherwise the robber wins. We analyzed several classes of graphs for this game in [9]:

* Tel.: +1 902 457 6310.
E-mail address: suzanne.seager@msvu.ca.

http://dx.doi.org/10.1016/j.tcs.2014.04.019

Theorem 1. *(See [9].) The cop wins on K_3 and $K_{2,3}$. However, the robber wins on any graph containing a copy of K_4 or $K_{3,3}$ as a subgraph.*

Theorem 2. *(See [9].) The cop wins on C_n for $n = 3$ or 4 and for all $n \neq 5$. However, the robber wins on C_5.*

Theorem 3. *(See [9].) The cop wins on any tree.*

We also provided bounds for the maximum number of probes required to locate the robber when the cop can win, some of which were improved by Brandt et al. [1].

Carraher et al. [4] have recently studied a relaxation of the robber locating game with the robber allowed to move to the probe vertex, producing several conjectures in the process. In this paper we provide counterexamples for two of those conjectures. Moreover, we prove that for this relaxation of the robber locating game, the robber wins on any graph containing a 6-cycle no edge of which is contained in an odd cycle, and that the cop wins on a tree if and only if the tree does not contain a vertex of degree at least three which is adjacent to three other vertices of degree at least three.

2. The backtrack robber locating game

In 2012, Carraher, Choi, Delcourt, Erickson, and West [4] introduced the *backtrack robber locating game*. At the start of this game, the robber chooses a vertex. The cop then chooses a *probe* vertex, and receives the distance from the probe to the robber. If the cop can locate the robber after this probe, she wins. Otherwise, the robber may move to an adjacent vertex or stay put. The cop then chooses a second probe, and so on. The cop wins if she can locate the robber after a finite number of probes; otherwise the robber wins.

Since this is a relaxation of the robber locating game, the robber can win in the backtrack game on every graph for which he can win in the original game. Thus, by Theorems 1 and 2, the robber can win on any graph containing a copy of K_4 or $K_{3,3}$ as a subgraph, and also on C_5.

For the rest of this paper we deal exclusively with the backtrack robber locating game. We first give some definitions. At the start of the cop's turn, the *extended robber set* is the set of all vertices where the robber could be located from the cop's point of view, so initially it is the entire vertex set. After the cop chooses a probe p and receives the distance d from the probe, the *robber set* for distance d is the set of all vertices where the robber could now be located; i.e., the set of all vertices in the extended robber set which are at distance d from p. Thus the cop wins at this point if and only if the robber set contains exactly one vertex. Otherwise, after the robber's turn to move, the next extended robber set will be the closed neighbourhood of the current robber set.

Consider the graph G_1 in Fig. 1. Table 1 summarizes a strategy for the cop to win on G_1. The first row of the table gives the first robber set $V(T)$ and the first probe, f. The distance from f to the robber must be 0, 1, 2, 3, 4, or 5, and the resulting robber set for each distance greater than 0 is listed (for $d = 0$ either the robber set contains the probe as its only vertex, or there is no robber set). If the robber set contains only one vertex, then the robber is located at that vertex. Otherwise, looking up the resulting robber set in the table gives the next probe, and so on. Moreover, the robber sets are ordered such that the robber sets for each distance after a probe all occur lower in the table than the robber set which was probed; in other words, the robber sets are ordered as a directed acyclic graph.

A *hideout* is a subgraph H of a graph G where the robber can win by remaining on the vertices of H. Carraher et al. [4] proved the following.

Theorem 4. *(See [4].) For any graph G, every cycle of length at most 5 is a hideout.*

A straight-forward adaptation of the proof of Theorem 2 establishes that the cop wins on C_n for $n > 6$. Carraher et al. [4] noted that the robber wins on C_6 but it remained open whether the robber wins on all graphs with girth 6. We have shown above that the graph G_1 is a counterexample. There is, however, an infinite class of graphs of girth 6 on which the robber wins.

Theorem 5. *Let G be a graph of girth 6 and let C be a 6-cycle in G such that no edge of C is contained in an odd cycle in G. Then C is a hideout of G.*

Proof. Let $C = v_1v_2v_3v_4v_5v_6$. We show that if at some point the extended robber set contains all but at most one vertex of C, then after the next probe p the robber is not located and there exists some distance from p such that the next extended robber set also contains all but at most one vertex of C. It suffices to show that there exists some distance from p such that the corresponding robber set contains at least two non-adjacent vertices of C, since the robber can then move to all but at most one vertex of C. Since the extended robber set initially contains all vertices of G, the result follows by induction.

So suppose that the extended robber set contains all vertices of $C - \{v_1\}$, say. Let p be the next probe, and let d be the minimum distance from p to a vertex of C. Suppose first that there exist two distinct vertices v_i and v_j of C both at