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Dense-choice Counter Machines revisited ☆

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A B S T R A C T

This paper clarifies the picture about Dense-choice Counter Machines (DCM), a less studied version of Counter Machines where counters range on a dense, rather than discrete, domain. The definition of DCM is revisited to make it extend (discrete) Counter Machines, and new undecidability and decidability results are proved. Using the first-order additive mixed theory of reals and integers, the paper presents a logical characterization of the sets of configurations reachable by reversal-bounded DCM. We also relate the DCM model to more common models of systems.

1. Introduction

Discrete (i.e. integer-valued) Counter Machines have been well-studied and still receive a lot of attention. We can mention Minsky Machines [2], different kinds of counter systems (e.g., [3,4], which are Minsky Machines using affine functions instead of increment/decrements and zero-tests, or [5,6]), Petri nets (or equivalently, VASS) and their many extensions.

There are also extensions of discrete counter systems to real-valued systems, called *hybrid* systems, such as linear hybrid automata, real-counter systems [7], or dense counter systems [8]. Another subclass of hybrid systems is the well-known decidable model of Timed Automata [9], which has been linked to special classes of counter systems in [5] and [6]. Recently, some connections between Timed Automata and timed Petri nets have been made [10,11]. An extension of counter systems to timed counter systems has been defined and studied in [12].

Reachability is already undecidable in linear hybrid automata [13], as well as in Timed Automata extended with only one stopwatch. Other subclasses of hybrid systems, like hybrid Petri nets (which includes stochastic, continuous, differential, and timed Petri nets) are *dense*, i.e. real-valued, extensions of Petri nets, but they have not the same semantics and their comparison is not always easy or feasible (see [14] for a recent survey).

From our point of view, the natural extension of (discrete) counter systems to dense counter systems is quite recent; to the best of our knowledge, the first paper which introduces Dense Counter Machines (DCM) as a natural generalization of Counter Machines (CM) is [8]. Their Dense Counter Machine allows incrementing/decrementing each counter by a real value δ chosen non-deterministically between 0 and 1. The motivation of this extension is to model hybrid systems where a non-deterministic choice can be made (see for example the argumentation about the dense producer/consumer in [8], which

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neither Timed Automata nor hybrid automata can model in an easy way). However, what can we earn from extending CM (which have the total expression power of computability) into DCM? Non-trivial problems will remain, of course, undecidable. The direction followed by [8] is to find subclasses of DCM for which the binary reachability is still computable, such as reversal-bounded DCM.

Our contributions. We first give a general definition of Counter Systems containing all the variations of the nature of counters, such as discrete, dense-choice, purely dense-choice, etc. We then revisit the definition of "Dense Counter Machines" [8] into *Dense-choice Counter Machines* (shortly, also, *DCM*) so that it is now simpler, more precise and formal, and also more clearly understandable as a natural extension of Minsky Machines. Shortly, a DCM is a finite-state machine augmented with dense-choice counters, which can assume only non-negative real values. At each step, every dense-choice counter can be incremented/decremented by 0, 1, or by a non-deterministically-chosen δ, $0 < \delta < 1$ (which may be different at each step). We assume w.l.o.g. that for a given step, the *same* δ is used for all the counters. This δ increment/decrement is the essential difference between dense-choice and discrete counters. A DCM can also test a counter x_i against 0 (either $x_i = 0$ or $x_i > 0$).

Since dense-choice counters are (trivially) more general than discrete counters, we also study the model of *purely-DCM*, i.e. DCM in which counters lose the ability to increment/decrement by 1. We show that the restriction to *bounded* purely-DCM (i.e., there exists a constant bound b such that each counter is bounded by b) still produces an undecidable control-state reachability problem (even with four 1-bounded purely dense-choice counters).

We then consider an effective (i.e. whose binary reachability is computable) class of DCM: reversal-bounded DCM [8]. In order to model hybrid systems more easily, we wish to introduce the ability for a counter to be tested against an integer k (instead of 0): this is an easy, common extension for Minsky Machines and for Petri nets, but it produces new technical problems for reversal-bounded DCM. One of the reasons is that the usual simulation of a k-test (i.e., several decrements and 0-tests, followed by increments restoring the original counter value), does not preserve reversal-boundedness. We actually show that reversal-bounded DCM with k-tests are equivalent to reversal-bounded DCM. This allows us to obtain as a corollary that the reachability relation of a DCM with one free counter and a finite number of reversal-bounded k-testable counters is still effectively definable by a so-called *mixed formula*, which is a formula of a decidable logic equivalent to the first-order additive mixed theory of reals and integers, $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, +, <)$. This extends a previous result of [8].

We give a logical characterization in $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, +, <)$ of the sets of configurations reachable by reversal-bounded DCM, and we prove that any mixed formula is the reachability relation of a reversal-bounded DCM. This completes the initial result stating that the reachability relation of a reversal-bounded DCM is definable by a mixed formula.

Finally, we relate DCM to other models of systems that are more common in verification. In particular, we show that DCM are more expressive than one of the most powerful models of counter automata, and also than the standard timed automata model.

Outline. This paper is divided into four main sections. Section 2 introduces a new formal definition of DCM, and presents different classes of DCM along with some of their properties. In Section 3, known and new decidability results for reachability problems are presented for several classes of DCM. In Section 4, the main theorem gives a full characterization of reversal-bounded DCM by mixed formulae. Finally, Section 5 compares DCM to other common models of systems.

2. Dense-choice Counter Machines

Notations. We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers, $\mathbb{Q}_+$ the set of non-negative rational numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{N}$ the naturals. Capital letters (e.g. X) denote sets, and small letters (e.g. x) denote elements of sets. Bold-faced symbols (e.g. $\mathbf{x}$) denote vectors, and subscripted symbols (e.g. x_i) denote components of vectors. Sometimes, for the sake of readability, we use x instead of x_i (without ambiguity). Throughout this paper, $n \in \mathbb{N}$ is the number of counters.

2.1. Extending Minsky Machines

In this section, we motivate the use of Dense-choice Counter Machines, by arguing about possible ways to extend Minsky Machines [2]. Minsky Machines are indeed the most elementary definition of Counter Systems that we will consider here, and probably the most known. A Minsky Machine has a finite set of control states, and operates transitions between them by executing instructions on a finite set of integer-valued variables (the counters). Its possible instructions are (1) increment a counter value by 1, (2) test if a counter value is 0, and (3) if a counter value is greater than 0 then decrement it by 1.

Let $\mathscr{L}$ be a given logic, such as the Presburger logic $\mathrm{FO}(\mathbb{N}, +, =)$, the mixed linear arithmetic $\mathrm{FO}(\mathbb{R}, \mathbb{Z}, +, <)$, etc. A formula $\mathcal{F}(\mathbf{x}, \mathbf{x}')$ of $\mathscr{L}$, with $2n$ free variables, is interpreted as the transformation of counter values $\mathbf{x}$ into $\mathbf{x}'$: it defines the counter values *before* and *after* the firing of a transition labelled by the binary relation $\mathcal{F}(\mathbf{x}, \mathbf{x}')$. Throughout this paper, we will use several different classes of counter machines, each one based on the generic Definition 2.1. They all use a finite labelling alphabet $\Sigma \subseteq \mathscr{L}$ defining instructions on a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The way the alphabet Σ is defined is what makes the difference between various Counter System classes.

Definition 2.1. A *Counter System* (CS for short) is a tuple $\mathscr{M} = \langle S, T \rangle$ such that S is a finite set of control states, and $T \subseteq S \times \Sigma \times S$ is a finite set of transitions.