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Approximation algorithm for the minimum weight connected k-subgraph cover problem ☆

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ABSTRACT

A subset F of vertices is called a connected k-subgraph cover (VCC_k) if every connected subgraph on k vertices contains at least one vertex from F. The minimum weight connected k-subgraph cover problem ($MWVCC_k$) has its background in the field of security and supervisory control. It is a generalization of the minimum weight vertex cover problem, and is related with the minimum weight k-path cover problem ($MWVCP_k$) which requires that every path on k vertices has at least one vertex from F. A k-approximation algorithm can be easily obtained by LP rounding method. Assuming that the girth of the graph is at least k, we reduce the approximation ratio to $k-1$, which is tight for our algorithm.

1. Introduction

The topology of a wireless sensor network (WSN) can be modeled as a graph, in which vertices represent sensors and edges represent communication channels between sensors. In recent years, new security protocols for WSN emerge. For example, in the k-generalized Canvas scheme [12] which guarantees data integrity, two kinds of sensor devices, protected and unprotected, are distinguished. An attacker is unable to copy data from a protected device. Suppose each information can be stored in a path of k vertices. So, it is required that every such a path has at least one protected vertex. The problem is to minimize the cost of the network by minimizing the number of protected vertices. Such a consideration leads to the *minimum weight k-path vertex cover problem* ($MWVCP_k$), the goal of which is to find a minimum weight vertex set F such that every path of k vertices contains at least one vertex from F.

In this paper, we propose a related problem as follows: Given a graph $G=(V,E)$ and a vertex-weight function w, the goal is to find a minimum weight vertex set $F\subseteq V$ such that every connected subgraph on k vertices has at least one vertex from F. Call such a set F as a *connected k-subgraph cover* (VCC_k) and the problem as a *minimum weight connected k-subgraph cover* problem ($MWVCC_k$). For $k=2$, $MWVCC_2$ is exactly the minimum weight vertex cover problem. For $k=3$, $MWVCC_3$ is the same as $MWVCP_3$.

This problem also has its background in the field of security and supervisory control. For example, in a WSN, if an attacker knows at least k related information fragments, then he can decode the whole information. Therefore, every connected k-vertex set must have at least one protected vertex to ensure security. For another example, if every k connected

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sensors can work as a group, then in order to control their work, at least one sensor from every potential work group should be supervised.

It is not difficult to obtain a k-approximation for $MWVCC_k$ (as well as for $MWVCP_k$), using LP rounding technique. Under the assumption that the girth (the length of a shortest cycle) is at least k, we improve the approximation ratio for $MWVCC_k$ to $k-1$. Factor $k-1$ is tight for our algorithm.

In [14], Tu and Zhou gave a 2-approximation algorithm for $MWVCP_3$. Since $MWVCP_3$ is the same as $MWVCC_3$ and the girth of a simple graph is always at least three, Tu and Zhou's result [14] is included in our result.

The remainder of this paper is organized as follows. We first introduce some related works in Section 2. In Section 3, we present our algorithm and its theoretical analysis. In Section 4, we conclude the paper with a discussion on future work.

2. Related work and preliminaries

Related work in this section is focused on approximation results on Minimum k-Path Vertex Cover problem ($MVCP_k$) and Minimum Weight k-Path Cover problem ($MWVCP_k$).

The $MVCP_k$ problem was proposed in [12]. In [5], Bresar et al. gave a polynomial-time approximation-preserving reduction from the Minimum Vertex Cover problem to $MVCP_k$, which, combining with [6], implies that for every $k \geq 2$, $MVCP_k$ is not able to be approximated within a factor of 1.3606 unless P = NP. They also gave a linear-time algorithm for $MVCP_k$ on trees and some upper bounds on the minimum cardinality of VCP_k. In [9], Kardoš et al. presented a polynomial-time randomized approximation algorithm for VCP_3 with an expected approximation ratio 23/11. They also formulated as an open problem whether $MVCP_k$ has a constant approximation for each $k \geq 2$. It was proved by Tu et al. [13] that $MVCP_3$ is NP-hard even for a cubic planar graph of girth 3, and a 1.57-approximation greedy algorithm was given for VCP_3 in cubic graphs. Recently, Li and Tu [10] presented a 2-approximation for VCP_4 in cubic graphs.

Requiring that the k-path vertex cover induces a connected subgraph, the problem is the Minimum k-Path Connected Vertex Cover ($MCVCP_k$). In [11], Liu et al. gave a PTAS for $MVCP_k$ in unit disk graphs.

The above works mainly concentrate on unweighted VCP_k problem.

Considering weight, Tu and Zhou [14] gave a 2-approximation for $MWVCP_3$ by using a layering method. By using a primal–dual method, they also achieved a 2-approximation [15].

For general k, it is not difficult to obtain a k-approximation for $MWVCP_k$ as well as for $MWVCC_k$. In fact, $MWVCC_k$ can be modeled as the following integer linear program:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} w_i x_i \\ \text{s.t.} \quad & \sum_{i \in S} x_i \geq 1, \quad \forall S \subseteq V,\ |S| = k,\ G[S] \text{ is connected}, \\ & x_i \in \{0,1\}, \quad i = 1, 2, \cdots, n, \end{aligned} \tag{1}$$

where $G[S]$ is the subgraph of G induced by vertex set S. By a classical rounding technique (see, for example [16]), one has a k-approximation. To be more concrete, solving the relaxed linear program of (1) (that is, relax $x_i \in \{0,1\}$ to $0 \leq x_i \leq 1$) to obtain an optimal fractional solution x^*. Let $x_i^A = 1$ if $x_i^* \geq \frac{1}{k}$, and $x_i^A = 0$ otherwise. Then $C = \{v_i : x_i^A = 1\}$ is a VCC_k of G and $w(C) = \sum_{i=1}^{n} w_i x_i^A \leq k \sum_{i=1}^{n} w_i x_i^* = k \cdot opt^f \leq k \cdot opt$, where opt^f is the optimal fractional value for the relaxation of (1) and opt is the optimal integral value for (1). The $MWVCP_k$ problem can be modeled by a similar 0–1 integer linear program as (1), except that "$G[S]$ is connected" is replaced by "$G[S]$ is a path on k vertices".

In this paper, we present a $(k-1)$-approximation for $MWVCC_k$ under the assumption that the girth of G is at least k, using local ratio method. Local ratio method was first proposed by Bar-Yehuda and Even [3], and has been used to design approximation algorithms for the feedback vertex set problem [1], the node deletion problem [8], resource allocation and scheduling problems [2], the minimum s–t cut problem and the assignment problem [4]. The readers may refer to [7] for a systematic introduction of the local ratio method. The key step in obtaining the desired approximation ratio is to find a special weight function w_1 and prove the desired approximation ratio with respect to w_1. In the following section, we shall put our focus on how to realize this step for $MWVCC_k$, and how to make use of such w_1 recursively.

3. The algorithm and its theoretical analysis

Let $d_G(v)$ denote the degree of vertex v in G. The subscript G is omitted if there is no ambiguity in the context. Given a vertex subset S, let $E[S]$ denote the set of edges having both ends in S, and let $G[S]$ denote the subgraph of G induced by S. Notice that $F \subseteq V$ is a VCC_k if every component of $G[V \setminus F]$ has cardinality at most $k-1$. A VCC_k F is said to be *minimal* if for any $v \in F$, $F - \{v\}$ is no longer a VCC_k. Let γ denote the size of a VCC_k of G with the smallest cardinality.

Theorem 3.1. *Let G be a connected graph, k be an integer with $k \geq 3$, and F be a VCC_k. Suppose the girth of G, denoted as $g(G)$, is at least k. Then*

$$\sum_{v \in F} (k-1) d(v) \geq (k-1)|E| - (k-2)|V| + (k-2)\gamma. \tag{2}$$