Theoretical Computer Science 535 (2014) 59–64

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Note

Computing degree and class degree

Mahsa Allahbakhshi [1]

Centro de Modelamiento Matematico, Universidad de Chile, Av. Blanco Encalada 2120, Piso 7, Santiago de Chile, Chile

ARTICLE INFO

Article history:
Received 27 March 2013
Received in revised form 5 March 2014
Accepted 2 April 2014
Communicated by D. Perrin

Keywords:
Factor codes
Sofic shifts
Measures of relative maximal entropy
Transition classes
Degree
Class degree

ABSTRACT

Let π be a factor code from a one dimensional shift of finite type X onto an irreducible sofic shift Y. If π is finite-to-one then the number of preimages of a typical point in Y is an invariant called the degree of π. In this paper we present an algorithm to compute this invariant. The generalized notion of the degree when π is not limited to finite-to-one factor codes, is called the class degree of π. The class degree of a code is defined to be the number of transition classes over a typical point of Y and is invariant under topological conjugacy. We show that the class degree is computable.

1. Introduction

One source of inspiration in symbolic dynamics comes from storage systems and transmission in computer science. For example sofic shifts are analogous to regular languages in automata theory, so a sofic shift and its cover are natural models for information storage and transmission. As a result, starting with a presentation of a dynamical system, there are known algorithms constructed to compute some kind of object from such presentation. Given a sofic shift, Coven and Paul constructed a finite procedure to obtain a finite-to-one sofic cover [5]. There is an algorithm to determine whether two graphs present the same sofic shift [8]. Kim and Roush showed that the shift equivalence of sofic systems is decidable [7].

In this work, starting from a sofic shift and its finite-to-one cover, we present an algorithm to compute the number of preimages of a typical point of the sofic. Moreover, we show that in the case of having an infinite-to-one cover, an analogous object can be computed in finitely many steps.

Given a factor code π from a one-dimensional shift of finite type X to a sofic shift Y, when π is finite-to-one there is a quantity assigned to π called the *degree* of π. The degree of a finite-to-one code is defined to be the minimal number of π-preimages of the points in Y. One can show that the number of preimages of every transitive point in Y is exactly the degree of π. The degree of a finite-to-one code is widely-studied and known to be invariant under recoding [8]. In the first section of this paper we present an algorithm to compute this invariant.

When $\pi : X \to Y$ is not limited to be finite-to-one an analogous of the degree, called the *class degree*, is defined to be the minimal number of transition classes (always finite) over the points in Y. The definition of a transition class is motivated by communicating classes in Markov chains. Roughly speaking, two preimages x and $\bar{x}$ of a point y in Y lie in the same equivalence class, *transition class*, if one can find a preimage z of y which is equal to x up to an arbitrarily large given

E-mail address: mallahbakhshi@dim.uchile.cl.

[1] The author was supported by FONDECYT project 3120137.

http://dx.doi.org/10.1016/j.tcs.2014.04.008
0304-3975/

positive coordinate and right asymptotic to $\bar{x}$ and vice versa. When π is finite-to-one then the degree and the class degree of π match. One can also show that the class degree is invariant under topological conjugacy and the number of transition classes over any transitive point of Y is exactly the class degree of π. One of the main applications of the class degree is bounding the number of measures of relative maximal entropy [1]. Such measures have applications in information theory, computing Hausdorff dimensions and functions of Markov chains [2–4,6,9]. In the second section of this paper we show that the class degree is computable.

2. Background

Throughout this paper, X is a one-dimensional shift of finite type (SFT) with the shift transformation T. The alphabet of X is denoted by $\mathcal{A}(X)$ and the σ-algebra on X generated by cylinder sets is denoted by $\mathscr{B}_X$. A triple (X, Y, π) is called a **factor triple** when $\pi : X \to Y$ is a continuous shift-commuting map (factor code) from an SFT X onto a subshift Y (sofic shift Y). When π is a one-block factor code induced by a symbol-to-symbol map $\pi_{\mathrm{b}} : \mathcal{A}(X) \to \mathcal{A}(Y)$ we naturally extend π_{b} to blocks in $\mathscr{B}_X$ (b stands for block). When π is a finite-to-one factor code there is a uniform upper bound on the number of pre-images of points in Y. The minimal number of pre-images of points in Y is called the **degree** of the code and is denoted by d_π.

Definition 2.1. We say two factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$ are **conjugate** if X is conjugate to $\tilde{X}$ under a conjugacy ϕ, Y is conjugate to $\tilde{Y}$ under a conjugacy ψ, and $\tilde{\pi} \circ \phi = \psi \circ \pi$.

Theorem 2.2. *(See [8].) Let (X, Y, π) be a factor triple. There is a factor triple $(\tilde{X}, \tilde{Y}, \tilde{\pi})$ conjugate to (X, Y, π) such that $\tilde{X}$ is one-step and $\tilde{\pi}$ is one-block.*

Theorem 2.3. *(See [8].) Given two conjugate factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$, we have $d_\pi = d_{\tilde{\pi}}$.*

Theorem 2.4. *(See [8].) Let π be a finite-to-one factor code from an SFT X onto an irreducible sofic shift Y. Then every transitive point of Y has exactly d_π preimages.*

Given a one-block factor code π, above every Y-block W there is a set of X-blocks U which are sent to W by π_{b}; i.e., $\pi_{\mathrm{b}}(U) = W$. Given $0 \leqslant i < |W|$, define

$$\pi_{\mathrm{b}}^{-1}(W)_i = \{a \in \mathcal{A}(X) \colon \exists W' \text{ with } \pi_{\mathrm{b}}(W') = W,\ W'_i = a\}$$

and

$$d^*_\pi = \min\{|\pi_{\mathrm{b}}^{-1}(W)_i| \colon W \in \mathscr{L}(Y),\ 0 \leqslant i < |W|\}.$$

Theorem 2.5. *(See [8].) Let π be a finite-to-one one-block factor code from an SFT X onto an irreducible sofic shift Y. Then $d^*_\pi = d_\pi$.*

Given a one-block factor code $\pi : X \to Y$, a **magic block** is a block W such that $d(W, i) = d^*_\pi$ for some $0 \leqslant i < |W|$. Such an index i is called a **magic coordinate** of W. A factor code π has a **magic symbol** if there is a magic block of π of length 1.

The class degree defined below is a quantity analogous to the degree which is defined in the general case when π is not only limited to be finite-to-one.

Definition 2.6. Suppose (X, Y, π) is a factor triple and $x, x' \in X$. There is a **transition** from x to x' denoted by $x \to x'$ if for each integer n, there is a point v in X so that

(1) $\pi(v) = \pi(x) = \pi(x')$, and
(2) $v_{-\infty}^n = x_{-\infty}^n$, $v_i^\infty = x_i'^\infty$ for some $i \geqslant n$.

Write $x \sim x'$, and say x and x' are in the same (equivalence) **transition class** if $x \to x'$ and $x' \to x$. The minimal number of transition classes over points of Y is called the **class degree** of π and denoted by c_π.

Theorem 2.7. *(See [1].) Given two conjugate factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$, we have $c_\pi = c_{\tilde{\pi}}$.*

Theorem 2.8. *(See [1].) Let π be a one-block factor code from a one-step SFT X to a sofic shift Y. The number of transition classes over a right transitive point of y is exactly the class degree.*

Theorem 2.10, in below, provides a finitary characterization of the class degree.