Theoretical Computer Science 531 (2014) 26–36

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

# Many-to-many two-disjoint path covers in restricted hypercube-like graphs

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ARTICLE INFO

*Article history:*
Received 16 July 2013
Received in revised form 14 January 2014
Accepted 14 February 2014
Communicated by S.-y. Hsieh

*Keywords:*
Disjoint path cover
Hypercube-like graph
Hamiltonian path
Hamiltonian cycle
RHL graph
Fault tolerance
Interconnection network

ABSTRACT

A Disjoint Path Cover (DPC for short) of a graph is a set of pairwise (internally) disjoint paths that altogether cover every vertex of the graph. Given a set $S$ of $k$ sources and a set $T$ of $k$ sinks, a many-to-many $k$-DPC between $S$ and $T$ is a disjoint path cover each of whose paths joins a pair of source and sink. It is classified as *paired* if each source of $S$ must be joined to a designated sink of $T$, or *unpaired* if there is no such constraint. In this paper, we show that every $m$-dimensional restricted hypercube-like graph with at most $m-3$ faulty vertices and/or edges being removed has a paired (and unpaired) 2-DPC joining arbitrary two sources and two sinks where $m \geqslant 5$. The bound $m-3$ on the number of faults is optimal for both paired and unpaired types.



## 1. Introduction

An interconnection network is frequently modeled as a graph in which the vertices and edges represent nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. One of the central issues in the study of interconnection networks is to detect (vertex-)disjoint paths, which is naturally related to routing among nodes and fault tolerance of the network [17,25].

Disjoint path is one of the fundamental notions in graph theory from which many properties of a graph can be deduced [2,25]. A *disjoint path cover* (DPC for short) of a graph is a set of pairwise (internally) disjoint paths that collectively cover every vertex of the graph. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1,27]. In addition, the problem is concerned with applications where full utilization of network nodes is important [32].

Let $G$ be an undirected graph. For a set of $k$ sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of $k$ sinks $T = \{t_1, t_2, \ldots, t_k\}$ such that $S \cap T = \emptyset$, a *many-to-many k-DPC* is a disjoint path cover composed of $k$ paths each of which joins a pair of source and sink. It partitions the vertex set $V(G)$ into $k$ subsets. The many-to-many $k$-DPC is called *paired* if each source $s_i$ should be joined to a specific sink $t_i$, whereas it is called *unpaired* if each source $s_i$ can be freely joined to a sink $t_j$ under an arbitrary bijection $\sigma$ from $S$ to $T$ where $t_j = \sigma(s_i)$. The other two possible $k$-disjoint path covers are of one-to-many type joining $S = \{s\}$ and $T = \{t_1, t_2, \ldots, t_k\}$, and of one-to-one type joining $S = \{s\}$ and $T = \{t\}$, which are clearly understandable. For more discussion, refer to [23,32].

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http://dx.doi.org/10.1016/j.tcs.2014.02.014

**Definition 1.** A graph $G$ is called *$f$-fault paired* (resp. *unpaired*) *$k$-disjoint path coverable* if $f + 2k \leqslant |V(G)|$ and $G$ has a paired (resp. unpaired) $k$-DPC joining an arbitrary set $S$ of $k$ sources and a set $T$ of $k$ sinks in $G \setminus F$ for any fault set $F$ where $S \cap T = \emptyset$ and $|F| \leqslant f$.

An $f$-fault *paired* $k$-disjoint path coverable graph is, by definition, $f$-fault *unpaired* $k$-disjoint path coverable. Given $S$ and $T$ in a graph $G$, it is NP-complete to determine if there exists a one-to-one, one-to-many, or many-to-many $k$-DPC joining $S$ and $T$ for any fixed $k \geqslant 1$ [32,33]. The disjoint path cover problems have been studied for graphs such as hypercubes [5–7,10,13,19,24], recursive circulants [20,21,32,33], and hypercube-like graphs [18,22,28,33], cube of a connected graph [29,30], and $k$-ary $n$-cubes [35,37]. Necessary conditions for a graph $G$ to be $f$-fault many-to-many $k$-disjoint path coverable have been established in terms of its connectivity $\kappa(G)$ and its minimum degree $\delta(G)$ [32,33], as shown below.

**Lemma 1.** (a) *If a graph $G$ with $|V(G)| \geqslant f + 2k + 1$ is $f$-fault* unpaired *$k$ ($\geqslant 2$)-disjoint path coverable, then $f + k \leqslant \delta(G) - 1$ [33].*
(b) *If a graph $G$ is $f$-fault* paired *$k$-disjoint path coverable, then $f + 2k \leqslant \kappa(G) + 1$ [32].*

Meanwhile, Restricted Hypercube-Like graphs (RHL graphs for short) [31] are a subset of nonbipartite hypercube-like graphs that have received much attention over the recent decades. For example, crossed cubes [12], Möbius cubes [8], twisted cubes [14], multiply twisted cubes [11], Mcubes [36], and generalized twisted cubes [4] are all RHL graphs. An $m$-dimensional RHL graph, which will be defined in the next section, has $2^m$ vertices. It is an $m$-regular graph of connectivity $m$.

Every $m$-dimensional RHL graph with $m \geqslant 3$ is known to be (a) $f$-fault *unpaired* $k$-disjoint path coverable for any $f$ and $k \geqslant 1$ subject to $f + k \leqslant m - 2$ [28], and (b) $f$-fault *paired* $k$-disjoint path coverable for any $f$ and $k \geqslant 2$ subject to $f + 2k \leqslant m$ [33]. The bound $m - 2$ on $f + k$ for the unpaired type and the bound $m$ on $f + 2k$ for the paired type respectively are one less than the optimal bounds of the necessary conditions of Lemma 1. It is still an open problem whether the optimal bounds can be achieved for all RHL graphs.

The problem has been partially solved in the sense that recursive circulants have the optimal bounds. Note that every odd-dimensional recursive circulant $G(2^m, 4)$ is included in RHL graphs (while not every even-dimensional recursive circulant is). Every $m$-dimensional recursive circulant $G(2^m, 4)$ with $m \geqslant 5$ is known to be (a) $f$-fault *unpaired* $k$-disjoint path coverable for any $f$ and $k \geqslant 2$ subject to $f + k \leqslant m - 1$ [20], and (b) $f$-fault *paired* $k$-disjoint path coverable for any $f$ and $k \geqslant 2$ subject to $f + 2k \leqslant m + 1$ [21].

In this paper, we achieve the optimal bounds of the necessary conditions of Lemma 1 for all RHL graphs where $k = 2$. In other words, we prove our main theorem that *every $m$-dimensional RHL graph is $(m - 3)$-fault* paired *2-disjoint path coverable where $m \geqslant 5$*. This leads to the fact that the graph is also $(m - 3)$-fault *unpaired* 2-disjoint path coverable. The bound $m - 3$ on the number of faults is the maximum possible for both paired and unpaired types.

Our contribution can also be seen as a generalization of fault-hamiltonicity of RHL-graphs, discovered in [31], that every $m$-dimensional RHL graph is $(m-3)$-fault hamiltonian-connected, where a graph is said to be *hamiltonian-connected* if every pair of vertices are joined by a hamiltonian path. Note that a paired (or unpaired) 2-disjoint path coverable graph is always hamiltonian-connected [32]. To be precise, a graph $G$ has a hamiltonian path from $s$ to $t$ passing through a prescribed edge $(x, y)$, where $\{x, y\} \cap \{s, t\} = \emptyset$ and $x$ is required to be visited before $y$, if and only if $G$ has a paired 2-DPC joining the $(s, x)$ and $(y, t)$ pairs (i.e., $s_1 = s$, $t_1 = x$, $s_2 = y$, and $t_2 = t$). If the order in which the two end-vertices of the prescribed edge $(x, y)$ are encountered during traversal of a hamiltonian path from $s$ to $t$ does not matter, it suffices to employ an unpaired 2-DPC joining $S = \{s, t\}$ and $T = \{x, y\}$ (instead of the paired one).

The rest of this paper is organized as follows. We give preliminaries in Section 2. Sections 3 and 4 are then devoted to a proof of our main theorem. Finally, we conclude in Section 5.

## 2. Preliminaries

A 3-dimensional RHL graph is isomorphic to recursive circulant $G(8, 4)$ that has a vertex set $\{v_i\colon\ 0 \leqslant i \leqslant 7\}$ and an edge set $\{(v_i, v_j)\colon\ i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$. The 3-dimensional RHL graph is also isomorphic to a 3-dimensional twisted cube $TQ_3$ or a Möbius ladder with four spokes [26] shown in Fig. 1. An $m$-dimensional RHL graph, $m \geqslant 4$, is recursively defined with a graph operation $\oplus$. Given two graphs $G_0$ and $G_1$ with the same number of vertices and a bijection $\phi$ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v))\colon\ v \in V(G_0)\}$. To simplify the notation, we often omit the bijection $\phi$ from $\oplus_\phi$ when it is clear in the context.

**Definition 2.** (See [31].) A graph that belongs to $RHL_m$ is called an *$m$-dimensional RHL graph* where

- $RHL_3 = \{G(8, 4)\}$, and
- $RHL_m = \{G_0 \oplus_\phi G_1\colon\ G_0, G_1 \in RHL_{m-1}, \phi \text{ is a bijection from } V(G_0) \text{ to } V(G_1)\}$ for $m \geqslant 4$.

Every $m$-dimensional RHL graph, $m \geqslant 3$, is nonbipartite and has $2^m$ vertices of degree $m$. It can be easily verified by induction on $m$ that the graph has no triangle (cycle of length three) and there exist at most two common neighbors for