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Note

Inefficiency of Nash Equilibrium for scheduling games with constrained jobs: A parametric analysis ☆

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ABSTRACT

In this paper, we revisit the inefficiency of Nash Equilibrium of scheduling games by considering the Price of Anarchy (PoA) as a function of r, which is the ratio between the maximum and minimum size of jobs. For the social costs of minimizing makespan and maximizing the minimum machine load of all machines, we obtain the PoA for two and three machines, and the bound is tight for any $r \geqslant 1$. Lower bounds on the PoA for general number of machines are also presented.

1. Introduction

Recently, scheduling games have received considerable attention in the operation research and theoretical computer science community. The problem arises from competition of selfish agents for common resource in the Internet and wireless networks. Due to the lack of centralized decision maker, each job can freely choose a machine where it will be processed. We assume that jobs are rational that each job wishes to minimize its own cost. The focus on the study of scheduling game is the existence and property of different kinds of stable schedules, among which the most famous one is *Nash Equilibrium* (*NE* for short). A schedule is called *NE* if no job will reduce its cost by moving to a different machine unilaterally.

Though the behavior of each job is influenced by individual costs, the performance of the whole system is measured by certain *social cost*. It is well known that in most situation *NE* are not optimal from this perspective due to lack of coordination. The inefficiency of *NE* can be measured by the *Price of Anarchy* (PoA for short). The PoA of an instance is defined as the ratio between the maximal social cost of a *NE* and the optimal social cost. The PoA of the game is the supremum value of the PoA of all instances.

The PoA of a game can help us to determine whether dictatorial control over the jobs' choice is considerable to improve the system performance. If the PoA is close to 1, then all *NE* are near optimal. Thus the intervention is totally unnecessary. If the PoA is very large, then the intervention, though itself may be costly, may still have some merit. However, since PoA of the game is a kind of worst-case measure, the inefficiency of *NE* is usually overestimated. For example, the PoA of an instance that the sizes of jobs are close to each other may be much smaller than the PoA of the instance that the sizes of jobs vary dramatically. In real applications, we are often confronted with only a subset of instances that have some certain

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properties which may be known in advance. It is of realistic meaning to determine that whether the PoA for these restricted instances will be smaller.

In this paper, we will study the inefficiency of *NE* of scheduling games with instances of certain property. We are given a set of jobs $\mathcal{J} = \{J_1, J_2, \dots, J_n\}$. The size of J_j is p_j, $j = 1, \dots, n$. The ratio between the maximum and minimum size of all jobs is no more than r, where r is the *stretch factor* of the instance. Without loss of generalization, we assume that $1 \leqslant p_j \leqslant r$ for any j. There are m identical machines $M_1, M_2, \dots, M_m$. Each job can choose an arbitrary machine, and the choices of all jobs determine a schedule. The load of a machine is the total size of jobs that choose it. The cost of a job in a schedule is the load of the machine which it chooses. Two kinds of social costs are considered. The first is minimizing makespan, i.e., the maximal load of all machines. The second is maximizing the minimum load of all machines.

For any schedule σ^A of $\mathcal{J}$, denote by $L_i^A(\mathcal{J})$ the load of $M_i, i = 1, \dots, m$. Let $C^A_{\max}(\mathcal{J}) = \max_{i=1,\dots,m} L_i^A(\mathcal{J})$ and $C^A_{\min}(\mathcal{J}) = \min_{i=1,\dots,m} L_i^A(\mathcal{J})$ be the maximum load and minimum load of all machines, respectively. Denote by $C^*_{\min}(\mathcal{J})$ and $C^{**}_{\max}(\mathcal{J})$ the optimal social costs of maximizing the minimum machine load and minimizing the makespan, respectively. We omit $\mathcal{J}$ if there is no confusion. Let $\mathcal{N}(\mathcal{J})$ be the set of all *NE* of $\mathcal{J}$. The PoA of the scheduling game with social cost of minimizing the makespan is

$$\sup_{\mathcal{J}} \sup_{\sigma^A \in \mathcal{N}(\mathcal{J})} \left\{ \frac{C^A_{\max}(\mathcal{J})}{C^{**}_{\max}(\mathcal{J})} \right\}.$$

The PoA of the scheduling game with social cost of maximizing the minimum machine load is

$$\sup_{\mathcal{J}} \sup_{\sigma^A \in \mathcal{N}(\mathcal{J})} \left\{ \frac{C^*_{\min}(\mathcal{J})}{C^A_{\min}(\mathcal{J})} \right\}.$$

Furthermore, we consider PoA as a function of the stretch factor r. In other words, the first supremum in the definition of PoA is taken over all instances $\mathcal{J}$ which satisfying $1 \leqslant p_j \leqslant r$. Clearly, PoA is a nondecreasing function of r by definition.

Related results: The inefficiency of *NE* of scheduling games was first studied in [12]. For the social cost of minimizing the makespan, the PoA is $\frac{2m}{m+1}$ [11,13]. The PoA of scheduling games on uniform machines was given in [4,8,5,7]. Here, a set of m machines is called *uniform*, if there exist m positive numbers $s_1, s_2, \dots, s_m$, such that the size of J_j would become $\frac{p_j}{s_i}$ when it chooses M_i. For the scheduling game with social cost of maximizing the minimum machine load, the PoA is 1.7 for any number of machines [3]. When the number of machines is small, the bound of $\frac{7}{4} - \frac{1}{4\lfloor \frac{m}{2} \rfloor}$ is better, and is tight when $2 \leqslant m \leqslant 7$ [3]. Corresponding results for uniform machines can be found in [14,6].

The PoA with respect to stretch factor has also been studied for several scheduling games. For the scheduling game on uniform machines with social cost of minimizing the total cost of all jobs, Berenbrink et al. [1] showed that the PoA is at most $4r$ when $1 \leqslant p_j \leqslant r$. Ferraioli and Ventre [9] carried a similar study on unrelated machines, where there exist mn positive numbers p_{ji}, $i = 1, \dots, m$, $j = 1, \dots, n$ such that the size of J_j become p_{ji} when it chooses M_i. Assume that $1 \leqslant p_{ji} \leqslant r$ for any i and j, they presented upper and lower bounds on the PoA of scheduling game with three kinds of social costs with respect to r. For the social cost of minimizing the total load of all machines, the PoA is exactly r. For the social cost of minimizing the total cost of all jobs, the PoA is at most $1 + mr$ and at least r. For the social cost of minimizing the total weighted cost of all jobs, the PoA is at most $\frac{1}{2}(r^2 + 2r + \sqrt{r^4 + 4r^3})$ and at least r^2. Here, the *weighted cost* of J_j is defined as the product of the load of M_i and p_{ji} when J_j choose the M_i. To the authors best knowledge, there are no studies considering PoA as a function of stretch factor with social cost of minimizing the makespan or maximizing the minimum machine load.

There are also studies in the literature that consider the PoA of certain scheduling game as a function of other parameters. For the scheduling game on m identical machines with social cost of minimizing the total cost of all jobs, Chen and Gurel [2] showed that the PoA lie in the interval of $[\rho - 1, \rho + 1]$, where $\rho = \frac{\sum_{j=1}^n p_j}{n \min_{1 \leqslant j \leqslant n} p_j}$.

Our results: In this paper, we study the PoA of scheduling games on identical machines with respective to stretch factor r. For the social cost of minimizing the makespan, the PoA for two and three identical machines are

$$\begin{cases} \dfrac{2r}{r+1}, & 1 \leqslant r < 2, \\ \dfrac{4}{3}, & r \geqslant 2, \end{cases}$$

and