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Total coloring of embedded graphs with maximum degree at least seven ☆

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ABSTRACT

A k-total-coloring of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is k-total-colorable if it admits a k-total-coloring. In this paper, it is proved that any graph G which can be embedded in a surface Σ of Euler characteristic $\chi(\Sigma) \geqslant 0$ is $(\Delta(G)+2)$-total-colorable if $\Delta(G) \geqslant 7$, where $\Delta(G)$ denotes the maximum degree of G.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let G be a graph. We use V, E, Δ and δ to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. *Surfaces* in this paper are compact, connected two manifolds without boundary. All embeddings considered in this paper are 2-*cell embeddings*.

A *k-total-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is *k-total-colorable* if it admits a k-total-coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G is a k-total-colorable. Clearly, $\chi''(G) \geqslant \Delta + 1$. Behzad and Vizing posed independently the famous conjecture, known as the *Total Coloring Conjecture* (**TCC**).

Conjecture. *For any graph G, $\Delta + 1 \leqslant \chi''(G) \leqslant \Delta + 2$.*

This conjecture was verified by Rosenfeld [7] and Vijayaditya [9] for $\Delta = 3$ and by Kostochka [4,5] for $\Delta \leqslant 5$. For planar graphs, the only case of **TCC** that still open is $\Delta = 6$. Borodin [1] confirmed **TCC** for planar graphs with $\Delta \geqslant 9$. By applying the Four Color Theorem, this result was extended to $\Delta \geqslant 8$ (see [3]). In 1999, Sanders and Zhao [8] improved it to $\Delta \geqslant 7$. In this paper, we strengthen this result and get the following theorem.

Theorem 1. *Let G be a graph embedded in a surface Σ of Euler characteristic $\chi(\Sigma) \geqslant 0$. If $\Delta \geqslant 7$, then $\chi''(G) \leqslant \Delta + 2$.*

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Theorem 1 also improves the result in [12] that $\chi''(G) = \Delta + 2$ if G is a graph which can be embedded in a surface Σ of Euler characteristic $\chi(\Sigma) \geqslant 0$ and $\Delta \geqslant 8$. Thus, for graphs embedded in a surface Σ of Euler characteristic $\chi(\Sigma) \geqslant 0$, the only open case of **TCC** is also $\Delta = 6$. Some related results can be found in [2,6,10,11].

2. The proof of Theorem 1

For convenience, we introduce the following notations. Let $G = (V, E, F)$ be an embedded graph, where F is the face set of G. For a vertex v of G, the *degree* $d(v)$ is the number of edges incident with v, and for a face f of G, the *degree* $d(f)$ is the length of the boundary walk of f, where each cut edge is counted twice. A k-*vertex*, k^--*vertex* or a k^+-*vertex* is a vertex of degree k, at most k or at least k, respectively. Similarly, we can define a k-*face*, k^--*face* and a k^+-*face*. A cycle of length 3 is called a *triangle*. We use $(v_1, v_2, \ldots, v_n)$-cycle (or -face) or $(d(v_1), d(v_2), \ldots, d(v_n))$-cycle (or -face) to denote a cycle (or face) whose boundary vertices are $v_1, v_2, \ldots, v_n$ in the clockwise order in G. Denote by $n_k(v)$ the number of k-vertices adjacent to the vertex v, by $n_k(f)$ the number of k-vertices incident with the face f, and by $f_k(v)$ the number of k-faces incident with the vertex v.

Now, we prove Theorem 1. In [12], it was proved $\chi''(G) \leqslant \Delta + 2$ for $\Delta \geqslant 8$. So we assume $\Delta = 7$ in the following. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1 which is embedded in a surface Σ of Euler characteristic $\chi(\Sigma) \geqslant 0$, and with $|V| + |E|$ as small as possible. We first show some lemmas. Note that in all figures of the paper, vertices marked $\bullet$ have no edges of G incident with them other than those shown and pair of vertices marked with $\circ$ can be connected to each other, unless stated otherwise.

Lemma 1. [8] *The graph G has the following properties*:

(1) $\delta(G) \geqslant 3$.
(2) *If* $uv \in E(G)$ *with* $d(v) \leqslant 4$, *then* $d(u) + d(v) \geqslant 10$.
(3) *There is no* $(4, 6, 7)$-*triangle and no* $(3, 7, 7)$-*triangle.*
(4) *If a* 5-*vertex* v *is incident with five triangles, then* v *is adjacent to at least four* 7-*vertices.*
(5) *Let* v *be a* 5-*vertex and* v_1 *be adjacent to* v. *If* v *and* v_1 *have at least two common neighbors, then at most one of them is a* 5-*vertex.*
(6) *Let* v *be a* 5-*vertex and* v_1 *be a* 6-*vertex adjacent to* v. *If* v *and* v_1 *have at least two common neighbors, then none of them is a* 5-*vertex.*
(7) *If* v *is a* 5-*vertex, then the configurations of Fig.* 1 *are reducible.*
(8) *Let* v *be a* 7-*vertex and* v_1 *be adjacent to* v. *If* v *and* v_1 *have at least two common neighbors, then* $d(v_1) \geqslant 5$.

The proof of Lemma 1 can be found in [8], so we omit the details here.

Lemma 2. *Let* v, u *be two* 5-*vertices. If* v *is adjacent to* u, *then there is at most one* 5-*vertex between* v *and* u *which is incident with five* 3-*triangles.*

The proof of Lemma 2 is a long, tedious case analysis, so we move it to Section 3. In the following, we will continue to prove Theorem 1.

By Euler's formula $|V| - |E| + |F| = \chi(\Sigma)$, we have

$$\sum_{v \in V} \bigl(d(v) - 4\bigr) + \sum_{f \in F} \bigl(d(f) - 4\bigr) = -4\bigl(|V| - |E| + |F|\bigr) = -4\chi(\Sigma) \leqslant 0. \tag{1}$$

We define $c(x)$ to be the initial charge. Let $c(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} c(x) = -4\chi(\Sigma) \leqslant 0$. In the following, we will assign a new charge denoted by $c'(x)$ to each $x \in V \cup F$ according to the discharging rules. Since our rules only move charges around and do not affect the sum, we have

$$\sum_{x \in V \cup F} c'(x) = \sum_{x \in V \cup F} c(x) = -4\chi(\Sigma) \leqslant 0. \tag{2}$$

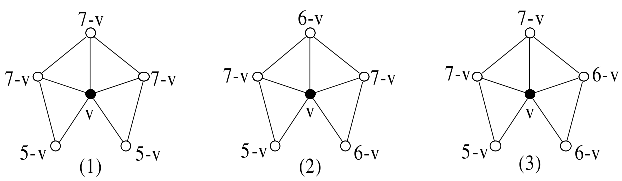

Fig. 1. Some reducible configurations.