Theoretical Computer Science 518 (2014) 124–127

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

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Note

Edge colorings of planar graphs without 5-cycles with two chords ☆

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A R T I C L E   I N F O

Article history:
Received 4 January 2013
Received in revised form 16 July 2013
Accepted 18 July 2013
Communicated by D.-Z. Du

Keywords:
Edge coloring
Planar graph
Cycle
Class 1

A B S T R A C T

A graph G is of class 1 if its edges can be colored with k colors in such a way that adjacent edges receive different colors, where k is the maximum degree of G. It is proved here that every planar graph is of class 1 if its maximum degree is at least 6 and any 5-cycle contains at most one chord.

1. Introduction

All graphs considered here are finite and simple. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. If $v \in V(G)$, then its neighbor set $N_G(v)$ (or simply $N(v)$) is the set of the vertices in G adjacent to v and the *degree* $d(v)$ of v is $|N_G(v)|$. We denote the maximum degree of G by $\Delta(G)$. For $V' \subseteq V(G)$, denote $N(V') = \bigcup_{u \in V'} N(u)$. A k-, k^+-vertex is a vertex of degree k, at least k. A k (or k^+)-vertex adjacent to a vertex x is called a k (or k^+)-neighbor of x. Let $d_k(x)$, $d_{k^+}(x)$ denote the number of k-neighbors, k^+-neighbors of x. A k-cycle is a cycle of length k. Two cycles sharing a common edge are said to be adjacent. Given a cycle C of length k in G, an edge $xy \in E(G)\setminus E(C)$ is called a *chord* of C if $x, y \in V(C)$. Such a cycle C is also called a chordal-k-cycle.

A graph is *k-edge-colorable*, if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The *edge chromatic number* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. In 1964, Vizing showed that if G is a graph with maximum degree Δ, then $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$. A graph G is said to be of *class* 1 if $\chi'(G) = \Delta$, and of *class* 2 if $\chi'(G) = \Delta + 1$. A graph G is *critical* if it is connected and of class 2, and $\chi'(G - e) < \chi'(G)$ for any edge e of G. A critical graph with maximum degree Δ is called a Δ-*critical graph*. It is clear that every critical graph is 2-connected.

For planar graphs, more is known. As noted by Vizing [2], if C_4, K_4, the octahedron, and the icosahedron have one edge subdivided each, class 2 planar graphs are produced for $\Delta \in \{2, 3, 4, 5\}$. He proved that every planar graph with $\Delta \geqslant 8$ is of class 1 (there are more general results, see [3] and [5]) and then conjectured that every planar graph with maximum degree 6 or 7 is of class 1. The case $\Delta = 7$ for the conjecture has been verified by Zhang [9] and, independently, by Sanders and Zhao [6]. The case $\Delta = 6$ remains open, but some partial results are obtained. Theorem 16.3 [2] stated that a planar

☆ This work was partially supported by National Natural Science Foundation of China (No. 11271006).
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0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.07.027

graph with the maximum degree Δ and the girth g is of class 1 if $\Delta \geqslant 3$ and $g \geqslant 8$, or $\Delta \geqslant 4$ and $g \geqslant 5$, or $\Delta \geqslant 5$ and $g \geqslant 4$. Lam, Liu, Shiu and Wu [4] proved that a planar graph G is of class 1 if $\Delta \geqslant 6$ and any vertex is incident with at most one 3-cycle. Zhou [10] obtained that every planar graph with $\Delta \geqslant 6$ and without 4- or 5-cycles is of class 1. Bu and Wang [1] proved that every planar graph with $\Delta \geqslant 6$ and without chordal 5-cycles and chordal 6-cycles is of class 1. Wang and Chen [7] proved that every planar graph is of class 1 if $\Delta \geqslant 6$ and it does not contain a 5-cycle with a chord. In the paper, we shall improve the above result by proving that every planar graph with $\Delta = 6$ and without 5-cycles with two chords is of class 1. Recently, Wang and Xu [8] proved that every plane graph G with maximum degree 6 is edge 6-colorable if no vertex in G is incident with four faces of size 3.

2. The main result and its proof

To prove our result, we will introduce some known lemmas.

Lemma 1. *(See [6,9].) If G is a planar graph with $\Delta(G) \geqslant 7$, then G is of class* 1.

Lemma 2. *(Vizing's Adjacency Lemma [2]). Let G be a Δ-critical graph, and let u and v be adjacent vertices of G with $d(v) = k$.*

(a) *If $k < \Delta$, then u is adjacent to at least $\Delta - k + 1$ vertices of degree Δ;*
(b) *If $k = \Delta$, then u is adjacent to at least two vertices of degree Δ.*

From Vizing's Adjacency Lemma, it is easy to get the following corollary.

Corollary 3. *Let G be a Δ-critical graph. Then*

(a) *every vertex is adjacent to at most one 2-vertex and at least two Δ-vertices;*
(b) *the sum of the degree of any two adjacent vertices is at least $\Delta + 2$;*
(c) *if $uv \in E(G)$ and $d(u) + d(v) = \Delta + 2$, then every vertex of $N(\{u, v\}) \setminus \{u, v\}$ is a Δ-vertex.*

Lemma 4. *(See [9].) Let G be a Δ-critical graph, $uv \in E(G)$ and $d(u) + d(v) = \Delta + 2$. Then*

(a) *every vertex of $N(N(\{u, v\})) \setminus \{u, v\}$ is of degree at least $\Delta - 1$;*
(b) *if $d(u), d(v) < \Delta$, then every vertex of $N(N(\{u, v\})) \setminus \{u, v\}$ is a Δ-vertex.*

Lemma 5. *(See [6].) No Δ-critical graph has distinct vertices x, y, z such that x is adjacent to y and z, $d(z) < 2\Delta - d(x) - d(y) + 2$, and xz is in at least $d(x) + d(y) - \Delta - 2$ triangles not containing y.*

To be convenient, we give some definitions and notations on planar graphs. Let G be a plane graph and $F(G)$ the face set of G. A face of G is said to be *incident* with all edges and vertices in its boundary. Two faces sharing an edge e are said to be *adjacent* at e. The degree of a face f of G, denoted by $d_G(f)$, is the number of edges incident with f where each cut edge is counted twice. A k-, k^+-face is a face of degree k, at least k. A k-face of G is called an $(i_1, i_2, \dots, i_k)$-face if the vertices in its boundary are of degrees $i_1, i_2, \dots, i_k$ respectively. A 3-face is denoted by $[x, y, z]$ if it is incident with distinct vertices x, y, z and $d(x) \leqslant d(y) \leqslant d(z)$. For a vertex $v \in V(G)$, we denote by $f_k(v)$ the number of k-faces incident with v.

Theorem 6. *Let G be a planar graph with $\Delta \geqslant 6$. If any 5-cycle contains at most one chord, then G is of class* 1.

Proof. Suppose that G is a counterexample to our theorem with the minimum number of edges and suppose that G is embedded in the plane. Then G is a 6-critical graph by Lemma 1, and it is 2-connected and Lemma 2. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{x \in V(G)} \bigl(d(x) - 4\bigr) + \sum_{x \in F(G)} \bigl(d(x) - 4\bigr) = -8 < 0.$$

We define ch to be the initial charge. Let $ch(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} ch(x) < 0$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V \cup F$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum. If we can show that $ch'(x) \geqslant 0$ for each $x \in V \cup F$, then we get an obvious contradiction $0 \leqslant \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) < 0$, which completes our proof.

A 4-face $f = [w, v, x, y]$ is called *special* if $d(x) = 2$ and v, x, y form a 3-face. The discharging rules are defined as follows.

R1 Let v be a 2-vertex. If v is incident with at least one 5^+-face, then v receives 1 from any of its incident 5^+-face, $\frac{1}{2}$ from each adjacent vertex; Otherwise, if v is incident with a special 4-face f, then v receives $\frac{1}{3}$ from f, $\frac{5}{6}$ from each adjacent vertex; Otherwise v receives 1 from each adjacent vertex.