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Fundamental study

Maximal pattern complexity of two-dimensional words

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Abstract

The maximal pattern complexity of one-dimensional words has been studied in several papers [T. Kamae, L. Zamboni, Sequence entropy and the maximal pattern complexity of infinite words, Ergodic Theory Dynam. Systems 22(4) (2002) 1191–1199; T. Kamae, L. Zamboni, Maximal pattern complexity for discrete systems, Ergodic Theory Dynam. Systems 22(4) (2002) 1201–1214; T. Kamae, H. Rao, Pattern Complexity over ℓ letters, E. Comb. J., to appear; T. Kamae, Y.M. Xue, Two dimensional word with $2k$ maximal pattern complexity, Osaka J. Math. 41(2) (2004) 257–265]. We study the maximal pattern complexity $p_\alpha^*(k)$ of two-dimensional words α. A two-dimensional version of the notion of *strong recurrence* is introduced. It is shown that if α is strongly recurrent, then either α is doubly periodic or $p_\alpha^*(k) \geqslant 2k$ $(k = 1, 2, \ldots)$. Accordingly, we define a *two-dimensional pattern Sturmian word* as a strongly recurrent word α with $p_\alpha^*(k) = 2k$. Examples of pattern Sturmian words are given.

Keywords: Two-dimensional words; Maximal pattern complexity; Strong recurrence

1. Introduction

The study of complexity of words has a very long history. Especially the words with low complexity have raised special interest. Recently, the study of complexity of words has been extended in two different directions. One direction is to study the complexity of higher-dimensional words (cf. [1–5,12,7,15,16]). Another is to consider a new complexity, the so-called *maximal pattern complexity* [10,11,8]. However, in this paper we will combine the above two efforts. We study the maximal pattern complexity of two-dimensional words.

1.1. Maximal pattern complexity

Let A be a finite alphabet. An element $\alpha = \alpha_0\alpha_1\alpha_2 \cdots \in A^{\mathbb{N}}$, where $\mathbb{N} := \{0, 1, 2, \ldots\}$, is called a *one-sided word* over A if every letter of A appears in α.

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Let k be a positive integer. By a *k-window* τ, we mean a sequence of integers of length k with

$$0 = \tau(0) < \tau(1) < \tau(2) < \cdots < \tau(k-1).$$

The k-window $\tau = \{0, 1, \ldots, k-1\}$ is called the *k-block window*. For a k-window $\tau : 0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ and a word α, the word

$$\alpha[n+\tau] := \alpha_{n+\tau(0)}\alpha_{n+\tau(1)} \cdots \alpha_{n+\tau(k-1)}$$

is the pattern of α through the window τ at position n. We denote by $F_\alpha(\tau)$ the set of all patterns of α through the window τ, i.e.,

$$F_\alpha(\tau) := \{\alpha[n+\tau];\, n = 0, 1, 2, \ldots\}.$$

In particular, we denote $F_\alpha(k) := F_\alpha(\tau)$ for the k-block window τ.

The *maximal pattern complexity function* p^*_α for a word α is introduced by the first author together with Zamboni [10] as

$$p^*_\alpha(k) := \sup_\tau \sharp F_\alpha(\tau) \quad (k = 1, 2, 3, \ldots),$$

where the supremum is taken over all k-windows τ, while the classical complexity function p_α is defined as $p_\alpha(k) = \sharp F_\alpha(k)$.

A classical result by Morse and Hedlund says that

Theorem A (*Morse and Hedlund [13]*). *For a word α, the following statements are equivalent*:
(i) *α is eventually periodic*,
(ii) *$p_\alpha(k)$ is bounded in k*,
(iii) *$p_\alpha(k) < k+1$ for some $k = 1, 2, \ldots$.*

The following parallel result with respect to the maximal pattern complexity function is proved in [10].

Theorem B (*Kamae and Zamboni [10]*). *For a word α, the following statements are equivalent*:
(i) *α is eventually periodic*,
(ii) *$p^*_\alpha(k)$ is bounded in k*,
(iii) *$p^*_\alpha(k) < 2k$ for some $k = 1, 2, \ldots$.*

A word α with block complexity $p_\alpha(k) = k+1$ $(k = 1, 2, 3, \ldots)$ is known as a *Sturmian word* and is studied extensively (see for example Berthé [2] and the references therein). Naturally, a word α with maximal pattern complexity $p^*_\alpha(k) = 2k$ $(k = 1, 2, 3, \ldots)$ is called a *pattern Sturmian word*. It is interesting that the classical Sturmian words are also pattern Sturmian words, and the class of pattern Sturmian words is larger than the class of Sturmian words [10].

The pattern complexity of a word α with more than two letters has been investigate in [8]. Let 1_S stand for the indicator function of the set S. A word α over A is called *periodic by projection* if there exists S with $\emptyset \neq S \subsetneq A$ such that the word

$$1_S \circ \alpha := 1_S(\alpha_0)1_S(\alpha_1)1_S(\alpha_2) \cdots \in \{0, 1\}^{\mathbb{N}}$$

is eventually periodic. Note that if $\ell = 2$, then α is periodic by projection if and only if α is eventually periodic. It is shown that

Theorem C (*Kamae and Rao [8]*). *Let α be a word over ℓ letters with $\ell \geqslant 2$. If $p^*_\alpha(k) < \ell k$ holds for some $k = 1, 2, \ldots$, then α is periodic by projection.*

Accordingly, a word over ℓ letters is said to be a *pattern Sturmian word* if $p^*_\alpha(k) = \ell k$ and it is not periodic by projection.