Available online at www.sciencedirect.com
SCIENCE DIRECT
Theoretical Computer Science 359 (2006) 344–368
Theoretical Computer Science
www.elsevier.com/locate/tcs

Distance bounds of ε-points on hypersurfaces☆

Sonia Pérez-Díaz[a,*], Juana Sendra[b], J. Rafael Sendra[c]

[a] *Dpto de Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain*
[b] *Dpto de Matemáticas, Universidad Carlos III, E-28911 Madrid, Spain*
[c] *Dpto de Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain*

Received 23 May 2005; received in revised form 31 March 2006; accepted 5 May 2006

Communicated by V. Pan

Abstract

ε-Points were introduced by the authors (see [S. Pérez-Díaz, J.R. Sendra, J. Sendra, Parametrization of approximate algebraic curves by lines, Theoret. Comput. Sci. 315(2–3) (2004) 627–650 (Special issue); S. Pérez-Díaz, J.R. Sendra, J. Sendra, Parametrization of approximate algebraic surfaces by lines, Comput. Aided Geom. Design 22(2) (2005) 147–181; S. Pérez-Díaz, J.R. Sendra, J. Sendra, Distance properties of ε-points on algebraic curves, in: Series Mathematics and Visualization, Computational Methods for Algebraic Spline Surfaces, Springer, Berlin, 2005, pp. 45–61]) as a generalization of the notion of approximate root of a univariate polynomial. The notion of ε-point of an algebraic hypersurface is quite intuitive. It essentially consists in a point such that when substituted in the implicit equation of the hypersurface gives values of small module. Intuition says that an ε-point of a hypersurface is a point close to it. In this paper, we formally analyze this assertion giving bounds of the distance of the ε-point to the hypersurface. For this purpose, we introduce the notions of height, depth and weight of an ε-point. The height and the depth control when the distance bounds are valid, while the weight is involved in the bounds.

Keywords: ε-Points; Distance bounds; Hypersurfaces; Approximate algorithms

1. Introduction

From the early beginnings of computer algebra, the achievements in symbolic computation have been related to many mathematical disciplines like linear algebra (e.g. homomorphic methods, fraction free techniques, etc.), non-linear algebra (e.g. resultants, gcd, polynomial factorizations, Gröbner bases, etc.), analysis (e.g. integration, computing with transcendental functions, solving differential equations, etc.), algebraic geometry (e.g. singularities computation, implicitization and parametrization techniques, etc.), etc.

In consequence of this development, symbolic algorithms have been used in some applications like, for instance, in computer-aided geometric design (see [19,20]), providing exact answers when dealing with algorithmic questions on mathematical entities exactly given. This type of contributions have been, and are, important since they offer effective

☆ Authors partially supported by the Spanish "Ministerio de Educación y Ciencia" under the Project MTM2005-08690-C02-01 and by the "Dirección General de Universidades de la Consejería de Educación de la CAM y la Universidad de Alcalá" under the project CAM-UAH2005/053.
* Corresponding author. Tel.: +34918854984.
E-mail addresses: sonia.perez@uah.es (S. Pérez-Díaz), jsendra@math.uc3m.es (J. Sendra), rafael.sendra@uah.es (J.R. Sendra).

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2006.05.020

algorithmic solutions to applied problem, and indeed investigations in this direction constitute an active research branch of symbolic computation.

Nevertheless, in many practical applications, these symbolic approaches tend to be insufficient, since in practice most of data objects are given or become approximate. This fact implies that intrinsic mathematical properties of the original object may fail. This phenomenon has motivated an increasing interest of the research community, working on computational algebra and computational algebraic geometry, for the development of approximate algorithms; that is, algorithms that deal symbolically with mathematical inputs, that have suffered a modification. For instance, let us assume that we are dealing with an applied problem where one needs to factorize a polynomial, and in fact, because of the theory behind the experiment or the application, one knows that the output polynomial must be reducible. Now, say that because of errors in the measures, the data is perturbed and instead of getting the polynomial $f := x^2 - y^2$, which factors as $(x - y)(x + y)$, one gets $\bar{f} := 1.00001x^2 + 0.00002xy - 1.00001y^2 + 0.00001$ that is irreducible. Every symbolic factorization algorithm will answer that $\bar{f}$ is irreducible, however $\bar{f}$ can be expressed as

$$\bar{f} = (1.00001x - y)(x + 1.00001y) + 0.00001,$$

which is "almost" reducible. An approximate factorization algorithm (see e.g. [7]) may recognize the above decomposition, and outputs that $\bar{f}$ factors approximately as $(1.00001x - y)(x + 1.00001y)$.

In algebra, approximate algorithms have been developed for computing polynomial greatest common divisors (see e.g. [6,11,25]), for finding zeros of multivariate systems (see e.g. [6,12,14]), for factoring polynomials (see e.g. [7,16,24,29]), for the computation of Gröbner basis (see e.g. [23,31]), etc. In algebraic geometry, approximate algorithms for computing singularities can be found in [2,3,9]; for implicitizating rational parametrizations in [8,10]; for implicitization methods in [4,15,18,26,27], etc.

In this field an important, and usually hard, step is the error analysis of the algorithms. This analysis mostly consists in estimating how "close" the input and the output of the algorithm are. If one is working from an algebraic point of view, for instance with polynomial factorizations, this question may be approached by measuring relative errors of polynomials. However, when the objects are studied from the geometric point of view, the Euclidean metric has to be taken into account, for instance, by requiring that each geometric entity lies in the offset region of the other at some small distance (see Section 5 for further details).

A technique to guarantee that an algebraic hypersurface (in practice, an algebraic curve or surface) is within the offset region of another, is the use of ε-points (see Definition 1), and more precisely, metric properties of this type of points. ε-points were introduced by the authors (see [26,27]) as a generalization of the notion of approximate root of a univariate polynomial. The notion of ε-point of an algebraic hypersurface is quite intuitive. It essentially consists in a point such that when substituted in the implicit equation of the hypersurface gives values of small module. This type of points play an important role in some algorithmic processes in algebraic geometry as the approximate parametrization (see [26,27]).

Theoretical properties and algorithmic questions of ε-points have been studied by several authors for the univariate case. For instance, bound analysis of roots of univariate polynomials can be found in [5,22,24], formulae for separating small roots of univariate polynomials are given in [30], the problem of constructing univariate polynomials with exact roots at some specific ε-roots (see Section 2 for the notion of ε-root) is analyzed in [21], condition numbers of ε-roots are studied in [32], etc.

Intuition says that an ε-point of a hypersurface is a point close to it. To state formally this assertion, one need to estimate the distance of an ε-point to the hypersurface, for instance by giving bounds. In [28] bounds for the case of plane curves are provided. In this paper, beside the obvious advances from curves to hypersurfaces, we improve the bounds given in [28]. The particularization to curves of the bounds given here are sharper than those in [28], and describes better the phenomenon showing how the multiplicity is involved in the number of points being close to the ε-singularity. The main ideas allowing us to improve and to extend the bounds in [28] to hypersurfaces are the notions of height, depth and (local and global) weight of an ε-point.

The paper is structured as follows. In Section 2, we introduce the basic notions of the paper. Section 3 is devoted to the study of distance properties between ε-roots and exact roots of univariate polynomials over $\mathbb{C}$. Section 4 focuses on the general case of hypersurfaces. In this study we distinguish the cases of ε-singularities and simple ε-points. In addition, in Section 4 a joint experimental analysis, of the bounds given in Sections 3 and 4, is included. In Section 5 we show the connection of the problem with the use of offsets to error analysis of approximate algorithms in algebraic