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Nimbers are inevitable

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ABSTRACT

This article concerns the resolution of impartial combinatorial games, in particular games that can be split in sums of independent positions. We prove that in order to compute the outcome of a sum of independent positions, it is always more efficient to compute separately the nimber of at least one of the independent positions, rather than to develop directly the game tree of the sum. The concept of the nimber is therefore inevitable to accelerate the computation of impartial games, even when we only try to determine the winning or losing outcome of a starting position. We also describe algorithms to use nimbers efficiently and to conclude, we give a review of the results obtained on two impartial games: Sprouts and Cram.

1. Introduction

Combinatorial games are games where two players alternate turns, with perfect knowledge of the current state of the game, and where chance is not involved. We will restrain our discussion to *impartial* combinatorial games, meaning that from a given position, the same moves are available to both players.

Moreover, the theorems and algorithms of this article apply only to impartial combinatorial games in their *normal* version, where the first player who cannot move loses, and not to the *misère* version, where the first player who cannot move wins.[1]

In particular, we will focus our attention on *splittable* impartial games, in which some of the positions can be split in sum of independent positions. Our purpose is to *solve* these games, i.e. to find which player has a winning strategy and to compute it explicitly. In Section 2, we review some background notions on impartial games, illustrating them with the games of Sprouts and Cram, and in Section 3, we give some insight on the central concept of nimber.

In Section 4, we develop the main result of this article: we prove that nimbers are necessary when we try to compute the outcome of a splittable impartial game. In Section 5, we detail algorithms to use nimbers efficiently and finally, in Section 6, we present the results obtained on the games of Sprouts and Cram.

2. Background

2.1. Sprouts and Cram

The algorithms described in this paper can be applied to any impartial combinatorial game played in the normal version, and we have chosen two well-known games for our computations, Sprouts and Cram.

The game of Sprouts starts with a given number of spots drawn on a sheet of paper. The players alternate drawing a line between two spots (possibly the same spot), and add a spot anywhere on the line they drew. The lines cannot cross each other, and a given spot cannot be used in more than 3 lines (Fig. 1).

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1 The analysis of impartial games in misère version is much more complicated, notably because the methods described in this article cannot be applied.

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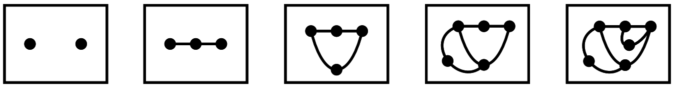

Fig. 1. Example of a Sprouts game, starting with 2 spots (the second player wins).

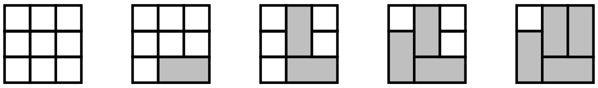

Fig. 2. Example of a Cram game on a 3 × 3 board (the second player wins).

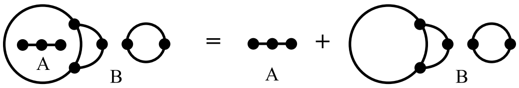

Fig. 3. Splittable Sprouts position.

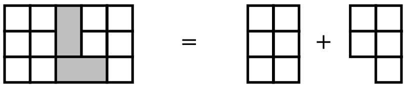

Fig. 4. Splittable Cram position.

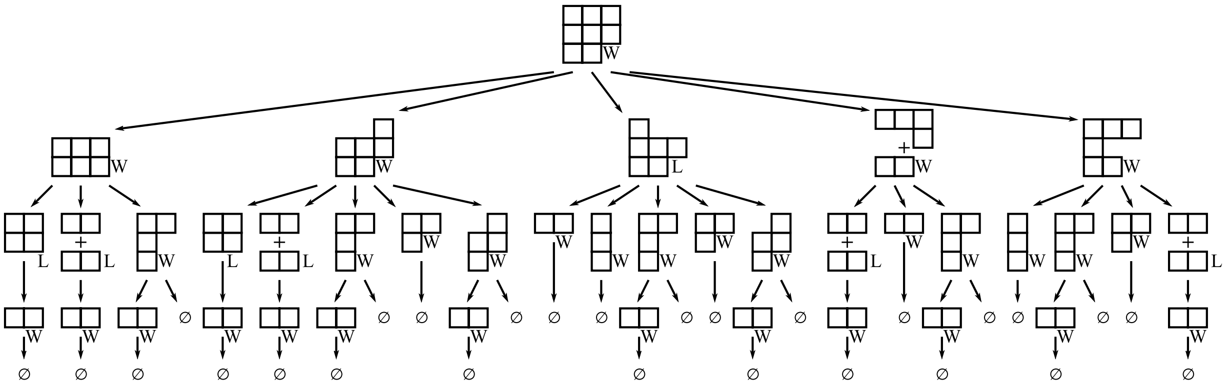

Fig. 5. Game tree of a Cram position.

The first article about Sprouts was written in 1967 by Gardner [1]. A detailed presentation of this game can be found in *Winning Ways* [2].

The game of Cram [3] is played on a board with very simple rules: players alternate filling two adjacent cells with a domino, until one of them cannot play anymore (Fig. 2). A description and an interesting analysis can be found in [2].

2.2. Splittable positions

Sprouts and Cram are *splittable* games, because some of the positions can be split into a sum of independent components. When a player moves in such a position, the move can only affect one of the components of the sum, leaving the others untouched.

For example, the position of Sprouts on Fig. 3 is splittable. The spots at the interface between regions A and B cannot be used anymore, and any further move must be done inside the region A (without affecting B) or inside the region B (without affecting A).

Fig. 4 gives another example. The position was obtained after playing two moves in a Cram game on a 3 × 5 board. The position is splittable, because the two components are independent.

2.3. Game tree

The *game tree of a position* $\mathscr{P}$ is the tree where nodes are the positions obtained by playing moves in $\mathscr{P}$, and in which two positions $\mathscr{P}_1$ and $\mathscr{P}_2$ are linked by an edge if $\mathscr{P}_2$ is an *option* of $\mathscr{P}_1$ (i.e. when $\mathscr{P}_2$ can be reached from $\mathscr{P}_1$ in one move).

The game tree of Fig. 5 has been obtained by identifying similar positions respectively to symmetry, and deleting isolated cells (since they cannot be used in any further move).