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Parallel addition in non-standard numeration systems

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ABSTRACT

We consider numeration systems where digits are integers and the base is an algebraic number β such that $|\beta| > 1$ and β satisfies a polynomial where one coefficient is dominant in a certain sense. For this class of bases β, we can find an alphabet of signed-digits on which addition is realizable by a parallel algorithm in constant time. This algorithm is a kind of generalization of the one of Avizienis. We also discuss the question of cardinality of the used alphabet, and we are able to modify our algorithm in order to work with a smaller alphabet. We then prove that β satisfies this dominance condition if and only if it has no conjugate of modulus 1. When the base β is the Golden Mean, we further refine the construction to obtain a parallel algorithm on the alphabet $\{-1, 0, 1\}$. This alphabet cannot be reduced any more.

1. Introduction

A positional numeration system is given by a base and by a set of digits. The base β is a real or complex number such that $|\beta| > 1$, and the digit set $\mathcal{A}$ is a finite alphabet of real or complex digits. The most studied numeration systems are of course the usual ones, where the base is a positive integer. But there have been also numerous studies where the base is an irrational real number (the so-called *β-expansions*), a complex number, or a non-integer rational number, *etc.* Some surveys can be found in [12, Chapter 7] and [8, Chapter 2].

In this work, the base β is an algebraic number, that is to say, the root of a polynomial with integer coefficients, and the digits of $\mathcal{A}$ are consecutive integers. We consider only finite β-representations, see Section 2 for definitions. Let us denote

$$\mathrm{Fin}_{\mathcal{A}}(\beta) = \left\{ \sum_{i \in I} x_i \beta^i \mid I \subset \mathbb{Z},\ \ I \text{ finite},\ \ x_i \in \mathcal{A} \right\}.$$

In general, the set $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is not closed under addition. In this paper, we will prove that when β is a root of a polynomial with a dominant coefficient, the alphabet $\mathcal{A}$ can be chosen in such a way that $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is a ring, and, moreover, addition can be performed by a parallel algorithm with time complexity $\mathcal{O}(1)$.

Addition of two numbers in the classical b-ary numeration system, where b is an integer $\geqslant 2$, has linear time complexity. In order to save time, several solutions have been proposed. A popular one is the Avizienis signed-digit representation [2], which consists of changing the digit set. Instead of taking digits from the canonical alphabet $\{0, \ldots, b-1\}$, they are taken from a symmetric alphabet of the form $\{-a, \ldots, 0, \ldots, a\}$, a being an integer such that $b/2 < a \leqslant b-1$ (b has to be $\geqslant 3$). Such a numeration system is *redundant*, that is to say, some numbers may have several representations. This property allows one to perform addition in constant time in parallel, because there is a limited carry propagation. A similar algorithm for base 2

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has been devised by Chow and Robertson [5], using alphabet $\{-1, 0, 1\}$. Here addition is realized in parallel with a window of size 3. Notice that Cauchy [4] already considered the redundant representation in base 10 and alphabet $\{-5, \ldots, 0, \ldots, 5\}$.

In symbolic dynamics, such functions computable in parallel are called *local*, more precisely p-local, which means that to determine the image of a word by a p-local function, it is enough to consider a sliding window of length p of the input. It is a general result that a p-local function is computable by an on-line finite automaton with delay $p-1$, see [6] for instance.

Amongst the non-standard bases, special attention has been paid to the complex ones, which allow to represent any complex number by a single sequence (finite or infinite) of natural digits, without separating the real and the imaginary part. For instance, it is known that every complex number can be expressed with base $-1+i$ and digit set $\{0, 1\}$, [14].

Parallel algorithms for addition in bases -2, $i\sqrt{2}$, $2i$ and $-1+i$ have been given in [13]. Results on addition in bases $-b$, $i\sqrt{b}$ and $-1+i$ in connection with automata theory have been presented in [6]. In particular, if b is an integer, $|b| \geqslant 3$, and $\mathcal{A} = \{-a, \ldots, 0, \ldots a\}$ with $a = \lceil \frac{|b|+1}{2} \rceil$, addition in base $\beta = \sqrt[q]{b}$ is computable in parallel and is a $(q+1)$-local function. If $|b| \geqslant 2$ is even, set $a = |b|/2$; then addition in base $\beta = \sqrt[q]{b}$ on $\mathcal{A}$ is a $(2q+1)$-local function.

The paper is organized as follows: First we suppose that the base β is a root of a polynomial with integer coefficients such that one of them is greater than twice the sum of the moduli of the other ones. We then say that β satisfies the *strong representation of zero property* (or, for short, that β *is SRZ*). We give a parallel algorithm, Algorithm I, for addition in that case, Theorem 3.2. When β is an integer $\geqslant 3$, it is the same algorithm as the one of Avizienis.

Section 4 is devoted to reduction of the working alphabet. We now suppose that β is a root of a polynomial such that one coefficient is greater than the sum of the moduli of the other ones. We then say that β satisfies the *weak representation of zero property* (or, for short, that β *is WRZ*). We give a parallel algorithm, Algorithm II, where the alphabet is reduced compared to Algorithm I, but there is a fixed number of iterations depending (only) on the weak polynomial satisfied by β, to be compared with Algorithm I which always has just one iteration.

We then show that in fact all algebraic numbers with no conjugate of modulus 1 are SRZ (or WRZ), and we give a constructive method to obtain the suitable polynomial (strong or weak representation of zero) from the minimal polynomial of β, Proposition 5.1.

In the end, we concentrate on the case where β is the Golden Mean. Algorithm II works with alphabet $\{-3, \ldots, 3\}$, which is big compared to the minimally redundant alphabet. Using ideas similar to the Chow and Robertson algorithm, we obtain a parallel algorithm on $\{-1, 0, 1\}$, Algorithm III. This algorithm can be applied to the Fibonacci numeration system as well.

2. Preliminaries

A finite *word* on the alphabet $\mathcal{A}$ is a concatenation of a finite number of letters of $\mathcal{A}$. The set of words on $\mathcal{A}$ is the free monoid $\mathcal{A}^*$. The set of bi-infinite words on $\mathcal{A}$ is denoted $\mathcal{A}^{\mathbb{Z}}$.

A finite β-*representation* of x with digits in $\mathcal{A}$ is a finite sequence $(x_i)_{m \leqslant i \leqslant n}$, with x_i in $\mathcal{A}$, such that $x = \sum_{i=m}^{n} x_i \beta^i$. It is denoted by the word

$$x_n x_{n-1} \cdots x_{m+1} x_m$$

with the most significant digit at the left hand side, as in the decimal numeration system.

The notion of a function computable in parallel comes from computer arithmetic (see [10]), where it is defined on finite words, but we give here a definition on bi-infinite words. A function $\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ is said to be *computable in parallel* if there exist two non-negative integers r and t, a positive integer k, and a function Φ from $\mathcal{A}^p$ to $\mathcal{B}^k$, with $p = r+t+k$, such that if $u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and $v = (v_i)_{i \in \mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$, then $v = \varphi(u)$ if, and only if, for every i in $\mathbb{Z}$, $v_{ki+k-1} \cdots v_{ki} = \Phi(u_{ki+k+t-1} \cdots u_{ki-r})$[1]. This means that the image of u by φ is obtained through a sliding window of length p. Such functions are computable in constant time in parallel.

The notion of a *local function* comes from symbolic dynamics (see [11]) and is often called a *sliding block code*. It is a function computable in parallel with $k = 1$. The parameter r is called the *memory* and the parameter t is called the *anticipation* of the function φ. The function φ is then said to be p-*local*.

To be self-contained, we recall the classical algorithms for parallel addition of Avizienis [2], and of Chow and Robertson [5]. In what follows, "for each i in parallel" means that "each numbered step is executed in parallel, and the results of the parallel computations are shared between the steps".

Algorithm of Avizienis: Base $\beta = b$, $b \geqslant 3$ integer, parallel addition on alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $b/2 < a \leqslant b-1$.

Input: two words $x_n \cdots x_m$ and $y_n \cdots y_m$ of $\mathcal{A}^*$, with $m \leqslant n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: a word $z_{n+1} \cdots z_m$ of $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m}^{n+1} z_i \beta^i.$$

[1] Careful! Indices of $\mathbb{Z}$ are decreasing from left to right.