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Paired many-to-many disjoint path covers in restricted hypercube-like graphs

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ABSTRACT

Given two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in a graph, a *paired many-to-many k-disjoint path cover* between S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which each path P_i runs from s_i to t_i. A family of hypercube-like interconnection networks, called *restricted hypercube-like graphs*, includes most non-bipartite hypercube-like networks found in the literature, such as twisted cubes, crossed cubes, Möbius cubes, recursive circulant $G(2^m, 4)$ of odd m, etc. In this paper, we show that every m-dimensional restricted hypercube-like graph, $m \geq 5$, with at most f vertex and/or edge faults being removed has a paired many-to-many k-disjoint path cover between arbitrary disjoint sets S and T of size k each, subject to $k \geq 2$ and $f + 2k \leq m + 1$. The bound $m + 1$ on $f + 2k$ is the best possible.

1. Introduction

One of the central issues in the study of interconnection networks is to detect vertex-disjoint paths, which are naturally related to routing among nodes and fault tolerance of the network [12,22]. Moreover, the disjoint path is one of the fundamental notions in graph theory, from which many properties of a graph can be deduced [2,22]. An interconnection network is frequently represented as a graph, in which the vertices and edges correspond to nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance.

Let G be a simple undirected graph whose vertex and edge sets, respectively, are denoted by $V(G)$ and $E(G)$. A *path cover* of G is a set of paths in G, such that every vertex of G is contained in at least one path. A *vertex-disjoint path cover*, or simply a *disjoint path cover*, of G is a special kind of path cover in which every vertex of G is covered by exactly one path. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1,23]. In addition, the problem is concerned with applications where full utilization of network nodes is important [30].

For a positive integer k, let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ be two disjoint subsets of $V(G)$. Then, a disjoint path cover $\{P_1, \ldots, P_k\}$ of G is said to be a *paired many-to-many k-disjoint path cover* (*paired k-DPC* for short) between S and T if P_i is a path that runs from s_i to t_i for all i. The disjoint path cover is said to be an *unpaired many-to-many k-disjoint path cover* (*unpaired k-DPC* for short) if for some permutation σ on $\{1, \ldots, k\}$, P_i runs from s_i to $t_{\sigma(i)}$ for all i [30]. Refer to Fig. 1 for examples of paired and unpaired DPCs. Note that a paired k-DPC joining S and T is, by definition, an unpaired k-DPC joining them. Here, the vertices in S and T are often called *sources* and *sinks*, respectively, and *terminals* collectively.

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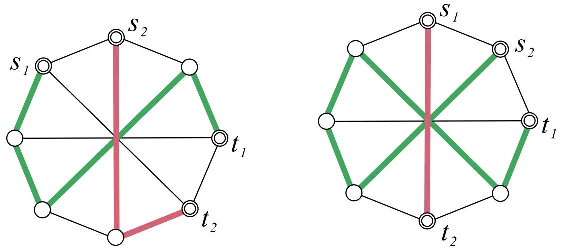

(a) Paired 2-DPC (b) Unpaired 2-DPC

Fig. 1. Examples of paired and unpaired DPCs. The configuration (b) admits no paired 2-DPC.

Definition 1. (See [31].) A graph G is called *f-fault paired* (resp. *unpaired*) *k-disjoint path coverable* if $f + 2k \le |V(G)|$ and G has a paired (resp. unpaired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G \setminus F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \le f$.

Simpler variants of the many-to-many disjoint path covers have also been investigated in previous literature. The *one-to-many k-DPC* for $S = \{s\}$ and $T = \{t_1, \dots, t_k\}$ is a disjoint path cover made of k paths, each joining a pair of source s and sink t_j, $j \in \{1, \dots, k\}$. The *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$ is a disjoint path cover, each of whose paths joins an identical pair of source s and sink t. The paths in the one-to-many k-DPC or in the one-to-one k-DPC may share a source and/or a sink and thus are pairwise internally disjoint. Readers are recommended to refer to the related literature, such as [15,19,26,30], for more details.

In this paper, we deal with the paired many-to-many disjoint path coverability of Restricted Hypercube-Like graphs (RHL graphs for short) [29], which are a subset of non-bipartite hypercube-like graphs that have received much attention over the past few decades. The class includes most non-bipartite hypercube-like networks found in the literature, as the following examples: twisted cubes [11], crossed cubes [9], Möbius cubes [6], recursive circulant $G(2^m, 4)$ of odd m [25], multiply twisted cubes [8], Mcubes [33], and generalized twisted cubes [3]. An m-dimensional RHL graph (whose definition is deferred to the next section) has 2^m vertices of degree m. Its connectivity is also m. The paired many-to-many disjoint path coverability for RHL graphs has been studied, as summarized in Theorem 1.

Theorem 1. *(See [17,30,31].)*

(a) Every m-dimensional RHL graph, $m \ge 3$, is f-fault paired k-disjoint path coverable for any f and $k \ge 1$ subject to $f + 2k \le m - 1$ [30].
(b) Every m-dimensional RHL graph, $m \ge 4$, is f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le m$ [31].
(c) Every m-dimensional RHL graph, $m \ge 5$, is $(m-3)$-fault paired 2-disjoint path coverable [17].

On the other hand, a necessary condition for a general graph G to be f-fault paired k-disjoint path coverable has been derived in terms of the connectivity $\kappa(G)$ of G in [30].

Lemma 1. *(See [30].) If a graph G is f-fault paired k-disjoint path coverable, then $f + 2k \le \kappa(G) + 1$.*

For the specific $k = 2$, the bound on $f + 2k$ in Theorem 1(c) is equal to that of Lemma 1, and is thus optimal. For general $k \ge 2$, however, the bound on $f + 2k$ was suggested to be $m - 1$ in Theorem 1(a) and then improved to be m in Theorem 1(b), still one less than the bound, $m+1$, of the necessary condition in Lemma 1. Bridging the gap in this paper, we will achieve the optimal bound $m + 1$ on $f + 2k$ for every $k \ge 3$. In other words, we will prove our main theorem asserting that *every m-dimensional RHL graph, $m \ge 5$, is f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le m + 1$.*

The rest of this paper is organized as follows: in the next section, we address previous works and definitions. Sections 3 and 4 are devoted to proving our main theorem. Finally, we conclude our findings in Section 5.

2. Previous works and definitions

Given the disjoint source set S and sink set T in a general graph G, it is NP-complete to determine if there exists a one-to-one, one-to-many, or many-to-many k-DPC between S and T for any fixed $k \ge 1$ [30,31]. The disjoint path cover problems have been studied for graphs such as hypercubes [4,5,7,10,14,21], recursive circulants [15,16,30,31], hypercube-like graphs [13,17,18,30,31], cubes of connected graphs [26,27], k-ary n-cubes [32,35], alternating group graphs [34], grid graphs [28], and tori [20]. In particular, the optimal construction of an unpaired k-DPC in an RHL graph has been established recently in [24], as shown in Theorem 2. Here, $\delta(G)$ denotes the minimum degree of a graph G.