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Structure connectivity and substructure connectivity of hypercubes

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ABSTRACT

The connectivity of a network – the minimum number of nodes whose removal will disconnect the network – is directly related to its reliability and fault tolerability, hence an important indicator of the network's robustness. In this paper, we extend the notion of connectivity by introducing two new kinds of connectivity, called structure connectivity and substructure connectivity, respectively. Let H be a certain particular connected subgraph of G. The H-structure connectivity of graph G, denoted $\kappa(G; H)$, is the cardinality of a minimal set of subgraphs $F = \{H'_1, H'_2, \ldots, H'_m\}$ in G, such that every $H'_i \in F$ is isomorphic to H, and F's removal will disconnect G. The H-substructure connectivity of graph G, denoted $\kappa^s(G; H)$, is the cardinality of a minimal set of subgraphs $F = \{J_1, J_2, \ldots, J_m\}$, such that every $J_i \in F$ is a connected subgraph of H, and F's removal will disconnect G. In this paper, we will establish both $\kappa(Q_n; H)$ and $\kappa^s(Q_n; H)$ for the hypercube Q_n and $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$.

1. Introduction

In the past decades, many regular-structured networks have been proposed as the model to interconnect processors in a multiprocessor computer system. Well-known examples include the mesh [18], the hypercube [10], several hypercube-variants such as the crossed cube [5] and the twisted cube [1], and the star graph [4]. The basic topology of an interconnection network can be modeled by a connected graph $G = (V, E)$, where V is the node set and E is the edge set. The nodes in V represent processors, and the edges in E represent the communication links between processors. In this paper, we use the terms *graph* and *network* interchangeably.

The *connectivity* is an important indicator of the reliability and fault tolerability of a network. The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum cardinality of a node set $F \subseteq V$, whose deletion would disconnect G or $G - F$ is a single node. However, this parameter has a deficiency. That is, it tacitly assumes that all nodes adjacent to the same node of G could fail at the same time, which is highly unlikely for large-scale systems. To compensate for this shortcoming, Harary [9] introduced the concept of *conditional connectivity*. Let G be a connected undirected graph, and let ρ be a graph-theoretic property. Harary defined the conditional connectivity $\kappa(G; \rho)$ as the minimum cardinality of a node set, if it exists, whose deletion disconnects G, but every remaining component would still have the property ρ.

Esfahanian and Hakimi [6,7] generalized the notion of connectivity by introducing the concept of *restricted connectivity*. A node set S is called a restricted node set if it contains no $N_G(x)$ as its subset for any node $x \in V(G)$. A restricted node set

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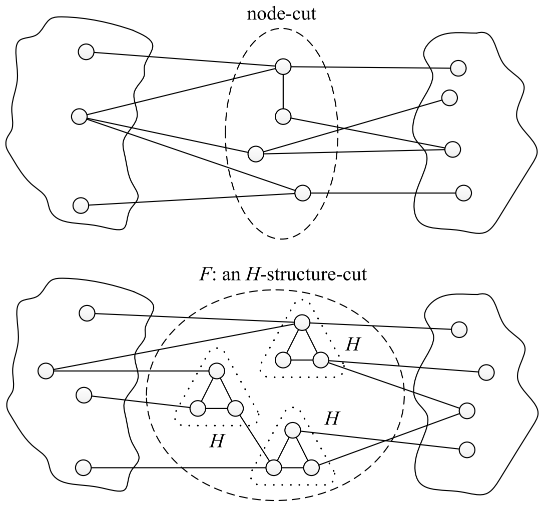

Fig. 1. Node-cut vs. H-structure-cut.

S is called a *restricted node cut* if $G - S$ is disconnected. The restricted connectivity of a connected graph G is defined as the minimum cardinality of a nontrivial node-set of G, if it exists. Compared with classic connectivity, restricted connectivity is more accurate to measure the reliability of the large multiprocessor system. Many works have been focused on the restricted connectivity of diverse graphs, such as vertex-transitive graphs [20] and three families of interconnection networks [3].

Following this idea, some new measures of connectivity were proposed and studied, including *g-extra connectivity* [8] and R_g*-connectivity* [17]. The g-extra connectivity of G, denoted by $\kappa_g(G)$, is the minimum cardinality of a node cut of G, whose deletion disconnects G, and every remaining component has at least $g + 1$ good nodes. Yang and Meng [21] determined g-extra connectivity of hypercubes. Later, Chang and Hsieh studied 2-extra connectivity and 3-extra connectivity of hypercube-like graphs [2].

An R_g-cut of a graph G is a node set, if any, whose deletion disconnects G and each node of the remaining components has at least g good neighbors. The R_g*-connectivity* of G, denoted by $\kappa^g(G)$, is the minimum cardinality of all R_g-cuts of G. Latifi [17] introduced the notion of R_g-connectivity and determined κ^g for hypercubes. The R_g-connectivity has been studied for other graphs like Cayley graphs [22] and (n, k)-star graphs [19] in recent years.

By the definitions of $\kappa(G)$, $\kappa_g(G)$, and $\kappa^g(G)$ (G may not be a complete graph), one can see that $\kappa(G) = \kappa_0(G) = \kappa^0(G)$. Therefore, the g-extra connectivity and the R_g-connectivity can be both regarded as a general form of the classical connectivity $\kappa(G)$. On the other hand, g-extra-connectivity and the R_g-connectivity also provide some more specific measures for the reliability and fault-tolerance of multiprocessor systems.

So far, most works on network reliability and fault-tolerance have focused on the effect of individual nodes becoming faulty. That is, the implied assumption is that the status of a node v – whether it is good or faulty – is an event *independent* of the status of nodes around v. However in reality, nodes that are linked could affect each other, and the neighbors of a faulty node might be more vulnerable and have a higher probability of becoming faulty later. Also to be noted is that increasingly in today's technology, networks and subnetworks are made into chips. That means if any node/nodes on the chip become faulty, the whole chip can be considered faulty. All these motivate the study of fault-tolerance not only from the perspective of individual nodes, but also based on some particular collection of linked nodes, or some *structure* of the network. Instead of considering the effect of nodes becoming faulty, we now look into the effect caused by some structures becoming faulty.

In this paper, we introduce two new kinds of connectivity, called *structure connectivity* and *substructure connectivity*, respectively. We say that a set F of connected subgraphs of G is a *subgraph-cut* of G if $G - V(F)$ is a disconnected or trivial graph. Let H be a connected subgraph of G. Then F is an H*-structure-cut* if F is a subgraph-cut, and every element in F is isomorphic to H. Fig. 1 shows an example of a node-cut vs. an H-structure-cut, where H is a 3-complete graph K_3. We define the H*-structure-connectivity* of G, denoted by $\kappa(G; H)$, to be the minimum cardinality of all H-structure-cuts of G.

We say that F is an H*-substructure-cut* if F is a subgraph-cut, such that every element in F is isomorphic to a connected subgraph of H. The H*-substructure-connectivity* of G, denoted by $\kappa^s(G; H)$, is the minimum cardinality of all H-substructure-cuts of G. Fig. 2 shows an example of an H-substructure-cut, where H is a 4-node cycle C_4. The subgraph-cut F contains 3 subgraphs, all of which are a connected subgraph of C_4.

In this paper, we will establish H-structure- and H-substructure-connectivity for $H = K_1$, $K_{1,1}$, $K_{1,2}$, $K_{1,3}$, and C_4 (shown in Fig. 3), respectively, in an n-dimensional hypercube. Note that K_1 is just a trivial single node. Therefore K_1-structure-connectivity reduces to the regular node connectivity, for which there are many known results.

The H-structure- and H-substructure-connectivity results in this paper are summarized as follows. We will show that for an n-dimensional hypercube Q_n, where $n \geq 4$,