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Towards tight bounds on theta-graphs: More is not always better ☆,☆☆

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ABSTRACT

We present improved upper and lower bounds on the spanning ratio of θ-graphs with at least six cones. Given a set of points in the plane, a θ-graph partitions the plane around each vertex u into m disjoint cones, each having aperture $\theta = 2\pi/m$, and adds an edge in each cone to the vertex whose projection onto the bisector of that cone is closest to u. We show that for any integer $k \geq 1$, θ-graphs with $4k+2$ cones have a spanning ratio of $1 + 2\sin(\theta/2)$ and we provide a matching lower bound, showing that this spanning ratio is tight.

Next, we show that for any integer $k \geq 1$, θ-graphs with $4k+4$ cones have spanning ratio at most $1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$. We also show that θ-graphs with $4k+3$ and $4k+5$ cones have spanning ratio at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. This is a significant improvement on all families of θ-graphs for which exact bounds are not known. For example, the spanning ratio of the θ-graph with 7 cones is decreased from at most 7.5625 to at most 3.5132. These spanning proofs also imply improved upper bounds on the competitiveness of the θ-routing algorithm. In particular, we show that the θ-routing algorithm is $(1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2)))$-competitive on θ-graphs with $4k+4$ cones and that this ratio is tight.

Finally, we present improved lower bounds on the spanning ratio of these graphs. Using these bounds, we provide a partial order on these families of θ-graphs. In particular, we show that θ-graphs with $4k+4$ cones have spanning ratio at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$, where θ is $2\pi/(4k+4)$. This is surprising since, for equal values of k, the spanning ratio of θ-graphs with $4k+4$ cones is greater than that of θ-graphs with $4k+2$ cones, showing that increasing the number of cones can make the spanning ratio worse.

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1. Introduction

A *geometric graph* G is a graph whose vertices are points in the plane and whose edges are line segments between pairs of points. A geometric graph G is called plane if no two edges intersect properly, i.e. any two edges can only intersect at their endpoints. Every edge is weighted by the Euclidean distance between its endpoints. The distance between two vertices u and v in G, denoted by $\delta_G(u, v)$, or simply $\delta(u, v)$ when G is clear from the context, is defined as the sum of the weights of the edges along the shortest path between u and v in G. A subgraph H of G is a t-spanner of G (for $t \geq 1$) if for each pair of vertices u and v, $\delta_H(u, v) \leq t \cdot \delta_G(u, v)$. The smallest value t for which H is a t-spanner is the *spanning ratio* or *stretch factor* of H. The graph G is referred to as the *underlying graph* of H. The spanning properties of various geometric graphs have been studied extensively in the literature (see [3,4] for a comprehensive overview of the topic).

This paper, and most of the literature, is concerned with the spanning ratio of classes (or families) of geometric graphs. The spanning ratio of a class of graphs is the supremum over the spanning ratios of all members of that graph class. Throughout the paper, we abuse terminology slightly by making statements such as "the spanning ratio of the $\theta_{(4k+x)}$-graph is at most $(\dots)$" when we refer to the spanning ratio of the family of $\theta_{(4k+x)}$-graphs. Similarly, we say that "the spanning ratio of the $\theta_{(4k+x)}$-graph is at least $(\dots)$", by which we mean that there exists a member of the family of $\theta_{(4k+x)}$-graphs whose spanning ratio is at least that.

Given a spanner, it is important to be able to route, i.e. find a short path, between any two vertices. A routing algorithm is said to be *c-competitive* with respect to G if the length of the path returned by the routing algorithm is not more than c times the length of the shortest path in G [5]. The smallest value c for which a routing algorithm is c-competitive with respect to G is the *routing ratio* of that routing algorithm.

In this paper, we consider the situation where the underlying graph G is a straightline embedding of the complete graph on a set of n points in the plane with the weight of an edge (u, v) being the Euclidean distance $|uv|$ between u and v. A spanner of such a graph is called a *geometric spanner*. We look at a specific type of geometric spanner: θ-graphs.

Introduced independently by Clarkson [6] and Keil [7], θ-graphs are constructed as follows (a more precise definition follows in Section 2): for each vertex u, we partition the plane into m disjoint cones with apex u, each having aperture $\theta = 2\pi/m$. When m cones are used, we denote the resulting θ-graph by the θ_m-graph. The θ-graph is constructed by, for each cone with apex u, connecting u to the vertex v whose projection onto the bisector of the cone is closest. Ruppert and Seidel [8] showed that the spanning ratio of these graphs is at most $1/(1 - 2\sin(\theta/2))$, when $\theta < \pi/3$, i.e. there are at least seven cones. This proof also showed that the *θ-routing* algorithm (defined in Section 2) is $1/(1 - 2\sin(\theta/2))$-competitive on these graphs.

Recently, Bonichon et al. [9] showed that the θ_6-graph has spanning ratio 2. This was done by dividing the cones into two sets, positive and negative cones, such that each positive cone is adjacent to two negative cones and vice versa. It was shown that when edges are added only in the positive cones, in which case the graph is called the half-θ_6-graph, the resulting graph is equivalent to the Delaunay triangulation where the empty region is an equilateral triangle. The spanning ratio of this graph is 2, as shown by Chew [10]. An alternative, inductive proof of the spanning ratio of the half-θ_6-graph was presented by Bose et al. [5], along with an optimal local competitive routing algorithm on the half-θ_6-graph.

It was also shown that the θ_5-graph is a spanner with spanning ratio at most $\sqrt{50 + 22\sqrt{5}} \approx 9.960$ [11] and the θ_4-graph is a spanner with spanning ratio at most $(1 + \sqrt{2}) \cdot (\sqrt{2} + 36) \cdot \sqrt{4 + 2\sqrt{2}} \approx 237$ [12]. Constructions similar to those for Yao-graphs [13] show that θ-graphs with fewer than 4 cones are not spanners. In fact, until recently it was not known that the θ_3-graph is connected [14].

Tight bounds on spanning ratios are notoriously hard to obtain. The standard Delaunay triangulation (where the empty region is a circle) is a good example. Its spanning ratio has been studied for over 20 years and the upper and lower bounds still do not match. Also, even though it was introduced about 25 years ago, the spanning ratio of the θ_6-graph has only recently been shown to be finite and tight, making it the first and, until now, only θ-graph for which tight bounds are known.

In this paper, we improve on the existing upper bounds on the spanning ratio of all θ-graphs with at least six cones. First, we generalize the spanning proof of the half-θ_6-graph given by Bose et al. [5] to a large family of θ-graphs: the $\theta_{(4k+2)}$-graph, where $k \geq 1$ is an integer. We show that the $\theta_{(4k+2)}$-graph has a tight spanning ratio of $1 + 2\sin(\theta/2)$.

We continue by looking at upper bounds on the spanning ratio of the other three families of θ-graphs: the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph, where k is an integer and at least 1. We use $4k + 4$ and $4k + 5$ in order to ensure that all bounds hold starting from $k = 1$, i.e. when there are at least 6 cones, the smallest value for which our inductive arguments work. We show that the $\theta_{(4k+4)}$-graph has a spanning ratio of at most $1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$. We also show that the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph have spanning ratio at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. As was the case for Ruppert and Seidel, the structure of these spanning proofs implies that the upper bounds also apply to the competitiveness of θ-routing on these graphs. These results are summarized in Table 1.

Finally, we present improved lower bounds on the spanning ratio of these graphs and we provide a partial order on these families. In particular, we show that θ-graphs with $4k + 4$ cones have spanning ratio at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$. This is surprising since, for equal values of k, the spanning ratio of θ-graphs with $4k + 4$ cones is greater than that of θ-graphs with $4k + 2$ cones, showing that increasing the number of cones can make the spanning ratio worse.

This paper is organized as follows. Section 2 precisely defines θ-graphs and introduces relevant notation. The proof of the new upper bounds on the spanning ratio begins in Section 3 with several key lemmas (in particular Lemmas 3 and 4).