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Evaluation of matrix algebra methods for calculating transformities from ecological and economic network data

Murray Patterson*

School of People, Environment and Planning, Massey University, Palmerston North, New Zealand

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ABSTRACT

This paper describes and characterises matrix algebra methods for calculating transformities from ecological and economic network data. Particular attention is given to those characteristics of complicated energy-flow and mass-flow networks that lead to the following methodological problems: joint production (co-products), non-square matrices (unequal number of processes and quantities), matrix singularity, problematic occurrence of negative transformities in the solution vector, and unequal emergy efficiencies which frequently occur in non-square matrices. Each of these problems is discussed and the means of resolving these problems are consequently presented. In addressing these problems, besides covering the use of the previously reported matrix inversion, regression, eigenvalue–eigenvector and singular value decomposition methods; a new method – the reflexive method – is introduced.

1. Introduction

Odum developed the concept of energy quality as it applies to complex systems, starting with his landmark book *Environment, Power and Society* (Odum, 1971). Odum's early work saw energy quality as being a measure of how much energy it takes to directly and indirectly produce a specified energy output. As pointed out by Sciubba (2010), this is a 'path function' measure of energy quality, as it is based on interdependencies (backward linkages) in the system of interest, in contrast to thermodynamic measures of energy quality that are based on 'point functions'. In this regard, Odum's first attempt to measure energy quality, involved first of all constructing straight chains of energy transformations based on transformations observed from the system of interest; then measuring the (first law) efficiency of these transformations; and then finally normalising this energy efficiency data using "fossil fuels" as the numeraire.

Odum in the following years developed more robust and comprehensive methods for measuring energy quality in complex economic and ecological systems. This culminated in the publication of the "Emergy Algebra"[1] procedures in Odum's (1996) book *Environmental Accounting*. In addition, in this book he incidentally reviewed and critiqued other methods of measuring energy quality. In the same year Brown and Herendeen (1996) outlined the four "rules"[2] of Emergy Algebra in a more concrete and succinct way:

1. All source emergy to a process is assigned to the processes' output.
2. By-products from a process have the total emergy assigned to each pathway.
3. When a pathway splits, the emergy is assigned to each leg of the split, based on its percent of total energy flow on a pathway.
4. Emergy cannot be counted twice within a system: (a) emergy in feedbacks cannot be double counted; (b) by-products, when reunited, cannot be added to equal a sum greater than the source emergy from which they were derived.

Odum (1996, pp. 100–101) provides a very good numerical example of the operationalisation of these rules which encapsulates most, if not all, of the problems encountered in measuring energy quality in complex systems; most notably how to deal with by-products (co-products) and the risk of double counting. Odum's numerical example uses the 'track summing' algorithm developed by Tennenbaum (1988).

Contemporaneously Patterson (1983, 1987, 1993) independently developed various mathematical methods for measuring

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* Correspondence address: Private Bag 11222, Palmerston North, New Zealand. Tel.: +64 6 3569099.

E-mail address: m.g.patterson@massey.ac.nz

[1] Of note, in the beginning of Chapter 6, Odum (1996) introduces the subject by using the word *algebra* in double quote marks. This seemingly indicates that it is not algebra in the formal or strict sense of the word. This interpretation, is reinforced by Odum's (1996) use of double quote marks elsewhere in his book, to indicate that the word is being used loosely or informally – e.g., on page 164, Odum writes "non-renewable" which seemingly indicates that he is using the word *non-renewable* not in a strict sense of the word.

[2] Of note, Odum (1996) himself did not use the word "rule". The word "rule" seems to have been introduced into the lexicon by Brown and Herendeen (1996) and since then has been frequently used by EMERGY proponents.

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energy quality in economic systems, based on solving simultaneous equations. Patterson (1983) called this the 'Quality Equivalent Method' which involved calculating 'quality coefficients' of each energy form (similar to the transformities) with these quality coefficients dependent on defining a 'quality equivalent unit' as a numeraire (similar to Odum's concept of EMERGY). Odum, 2002 (Odum, 2000; Collins and Odum 2000) considered the 'Quality Equivalent Methodology' to be a method for calculating transformities that relied on matrix algebra methods, as opposed to other methods such as 'track-summing' which do not use formal matrix algebra. Similar assessments of the 'Quality Equivalent Methodology' are made by Bartelmus (2003) and Hau and Basksi (2004). Patterson (2012), although broadly agreeing with this argument, however pinpoints a number of key differences between the 'Quality Equivalent Methodology' and the EMERGY method.

The purpose of this paper therefore is to critically review the formal *matrix algebra methods* for calculating 'transformities' including the matrix-based regression approach, input–output analysis/matrix inversion, eigenvalue–eigenvector method, singular value decomposition, and finally the hitherto unpublished reflexive method. In undertaking such a critical review, it is recognised, that there are other conceptual and numerical approaches to measuring 'transformities' in complex ecological and economic systems. These non-matrix algebra approaches are adequately explained and reviewed elsewhere (Odum, 1996; Bastianoni et al., 2011).

In undertaking this critical review, it is argued that formal matrix algebra approaches for calculating transformities have a number of strong advantages: (1) the background theory of these matrix algebra approaches are well-established in terms of their mathematical proofs, theorems and corollaries – hence, there is no need to debate the mathematical basis of these methods, which is sometimes necessary with non-matrix algebra methods of calculating transformities; (2) the matrix algebra methods have been widely applied in analysing complex economic and ecological networks, and therefore lessons learned from such applications can be readily transferred to transformities and other emergy-focussed calculations. For example, the problem of pricing/valuing by-products (co-production, joint production) has been thoroughly analysed by economists such as Pasinetti (1977) with direct implications for the calculation of transformities in complex systems; (3) from a practical point of view, the computational methods for calculating matrix-based transformities, have been thoroughly tested, evaluated and are now readily available in mathematical software packages such as *Matlab* and *Mathematica*; (4) with the possible exception of the 'reflexive method', all of these matrix approaches to transformity calculation, are mathematically elegant (efficient) and mathematically rigourous, requiring only a few lines of programming code. Very complicated systems of energy flow with many processes, joint production, feed-backs and split flows can be analysed just as readily as smaller and simpler systems, requiring exactly the same mathematical operations and software code.[3]

There are however drawbacks and limitations to the matrix algebra approaches. First of all, they may not be able to be directly used in all systems of energy flow – for example, if the system is mathematically under-determined (in this case, more transformities to be calculated than energy flow equations), then the transformities cannot usually[4] be determined by matrix algebra. Second, for example, in standard matrix inversion methods, if two or more energy flow equations are proportional to each other, which can happen for example when there are co-products/by-products, then the matrix will be singular and therefore cannot be inverted in the solution process. The 'rule-based' (e.g., 'track summing') methods can avoid such difficulties, although this sometimes comes with the 'philosophical baggage' required for justifying the rationale behind these rules. Interestingly, Li et al. (2010) have attempted to integrate the so-called 'rules' of emergy algebra into the formal matrix algebra approaches, by "pre-conditioning" the network equations before using matrix algebra methods to determine the transformities.

2. Matrix algebra methods for calculating transformities

Since the early 1980s, various matrix algebra methods have been developed for calculating transformities in complicated ecological and economic networks. These methods involve calculating transformities by solving a system of simultaneous linear equations which map the flow of energy and mass in these networks. In this section of this paper, a brief review of these matrix algebra methods summarises and complements the more detailed coverage in a recent paper by Patterson (2012) published in this journal *Ecological Modelling*. Readers are therefore referred to that paper for a more in-depth mathematical coverage of these methods – except for the 'reflexive method' whose mathematical basis is described for the first time in Appendix A of this current paper.

2.1. Regression method

Patterson (1983) introduced a regression method for calculating transformities in energy flow networks, and illustrated the application of this method using data for the New Zealand economy. The method involves solving simultaneous linear equations that measure energy flows in any economic or ecological system:

$$\mathbf{W}\boldsymbol{\beta} = \mathbf{V}\boldsymbol{\beta} + \mathbf{e} \tag{1}$$

Subject to: $\boldsymbol{\beta} \neq \mathbf{0}$

where $\mathbf{W}$ = matrix ($n \times m$) quantifying m energy inputs into n processes, known; $\mathbf{V}$ = matrix ($n \times m$) quantifying m energy outputs from n processes, known; $\boldsymbol{\beta}$ = vector ($m \times 1$) of transformities[5] (quality equivalents units/energy units), to be solved; $\mathbf{e}$ = vector ($n \times 1$) of residuals (quality equivalent units), to be solved.

By defining $\mathbf{V} - \mathbf{W} = \mathbf{X}$ and then by rearrangement we have:

$$\mathbf{X}\boldsymbol{\beta} + \mathbf{e} = \mathbf{0} \tag{2}$$

This problem can be solved by setting any one of the transformities $\boldsymbol{\beta}$ to unity which then becomes the numeraire[6]:

$$\mathbf{X}^{*}\boldsymbol{\beta} + \mathbf{e} = \mathbf{y} \tag{3}$$

The $\mathbf{X}^{*}$ matrix has the numeraire column of $\mathbf{X}$ removed and 'transferred' (with a sign change) to the other side of the equation

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[3] From this author's experience of applying the rules of 'EMERGY Algebra' to large, structurally complex economic networks with enmeshed patterns of feedback, some of the rules of EMERGY algebra become impossible to apply in an unambiguous way – e.g., it can be difficult to know when by-products are 're-united' in such networks.

[4] All of the formal matrix algebra methods (Sections 2.1, 2.2 and 2.4–2.6) outlined in this paper cannot be used to solve under-determined systems of equations; with the *exception* of the linear optimisation method (Section 2.3). However, although the linear optimisation can be 'theoretically' used to solve under-determined systems of equations, this will result in zero values for some transformities, which is usually considered to be an unacceptable solution as this does not make conceptual sense.

[5] Patterson (1983) used the term 'quality coefficient' instead of 'transformity'. It can be argued that the two terms are the same (Collins and Odum, 2000).

[6] Odum typically used solar energy as the operational numeraire but throughout his long career also used various fossil fuels as the numeraire. Patterson (1987, 2012) has consistently argued that the actual choice numeraire is arbitrary, and that it's the *numerical relativities* between the transformities that is of *fundamental importance*.