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Two greedy consequences for maximum induced matchings

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A B S T R A C T

We prove that, for every integer d with $d \geq 3$, there is an approximation algorithm for the maximum induced matching problem restricted to $\{C_3, C_5\}$-free d-regular graphs with performance ratio $0.708\bar{3}d + 0.425$, which answers a question posed by Dabrowski et al. (Theoret. Comput. Sci. 478 (2013) 33–40). Furthermore, we show that every graph with m edges that is k-degenerate and of maximum degree at most d with $k < d$, has an induced matching with at least $m/((3k-1)d - k(k+1) + 1)$ edges.

1. Introduction

A set M of edges of a graph G is an *induced matching* of G if the set of vertices of G that are incident with the edges in M induces a 1-regular subgraph of G, or, equivalently, if M is an independent set of the square of the line graph of G. The *induced matching number* $\nu_s(G)$ of G is the maximum cardinality of an induced matching of G. Induced matchings were introduced by Stockmeyer and Vazirani [15] as a variant of classical matchings [11]. While classical matchings are structurally and algorithmically well understood [11], it is hard to find a maximum induced matching [2,12,15] and efficient algorithms are only known for special graph classes [1,3,5]. The problem to determine a maximum induced matching in a given graph, called Maximum Induced Matching for short, is even APX-complete for bipartite d-regular graphs for every $d \geq 3$ [5,7].

On the positive side, a natural greedy strategy applied to a d-regular graph G, which mimics the well-known greedy algorithm for the maximum independent set problem applied to the square of the line graph of G, produces an induced matching with at least $\frac{m(G)}{2d^2-2d+1}$ edges. Since every induced matching of a d-regular graph contains at most $\frac{m(G)}{2d-1}$ edges, this already yields an approximation algorithm for Maximum Induced Matching in d-regular graphs with performance ratio $d - \frac{1}{2} + \frac{1}{4d-2}$ as observed by Zito [18]. This was improved slightly by Duckworth et al. [7] who describe an approximation algorithm with asymptotic performance ratio $d-1$. The best known approximation algorithm for Maximum Induced Matching restricted to d-regular graphs for general d is due to Gotthilf and Lewenstein [9] who elegantly combine a greedy strategy with a local search algorithm to obtain a performance ratio of $0.75d + 0.15$. In [10] Joos et al. describe a linear time algorithm that finds an induced matching with at least $\frac{m(G)}{9}$ edges for a given 3-regular graph G, which yields an approximation algorithm for cubic graphs with performance ratio $\frac{9}{5}$.

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At the end of [5] Dabrowski et al. propose to study approximation algorithms for regular bipartite graphs, and to determine whether the above performance ratios can be improved in the bipartite case. As our main result we show that this is indeed possible.

Theorem 1. *For every integer d with $d \geq 3$, there is an approximation algorithm for* Maximum Induced Matching *restricted to $\{C_3, C_5\}$-free d-regular graphs with performance ratio $\frac{17}{24}d + \frac{17d}{48d-24} \leq 0.708\bar{3}d + 0.425$.*

Our proof of Theorem 1 builds on the approach of Gotthilf and Lewenstein [9], and all proofs are postponed to Section 2.

Our second result, which also relies on a greedy strategy, is a lower bound on the induced matching number of degenerate graphs. This result is related to recent bounds on the *strong chromatic index* $\chi'_s(G)$ of a graph G [8], which is defined as the minimum number of induced matchings into which the edge set of G can be partitioned. The most prominent conjecture concerning this notion was made by Erdős and Nešetřil in 1985 and states that the strong chromatic index of a graph G of maximum degree at most d is at most $\frac{5}{4}d^2$. The most significant progress towards this conjecture is due to Molloy and Reed [14] who proved $\chi'_s(G) \leq 1.998d^2$ provided that d is sufficiently large. Again a natural greedy edge coloring implies $\chi'_s(G) \leq 2d^2 - 2d + 1$.

Recall that a graph G is k-*degenerate* for some integer k, if every non-empty subgraph of G has a vertex of degree at most k. Recently, Chang and Narayanan [4] studied the strong chromatic index of 2-degenerate graphs and inspired the following results about the strong chromatic index of a k-degenerate graph G of maximum degree at most d with $k \leq d$:

$$\chi'_s(G) \leq \begin{cases} 10d - 10, & \text{if } k = 2 \text{ [4]} \\ 8d - 4, & \text{if } k = 2 \text{ [13]} \\ (4k-1)d - k(2k+1) + 1, & \text{[6]} \\ (4k-2)d - k(2k-1) + 1, & \text{[16]} \\ (4k-2)d - 2k^2 + 1, & \text{[17].} \end{cases} \tag{1}$$

The proofs of (1) all rely in some way on greedy colorings and (1) immediately implies that $\nu_s(G) \geq \frac{m(G)}{\chi'_s(G)} \geq \frac{m(G)}{4kd + O(k+d)}$ for the considered graphs. We show that the factor 4 can be reduced to 3.

Theorem 2. *If G is a k-degenerate graph of maximum degree at most d with $k < d$, then $\nu_s(G) \geq \frac{m(G)}{(3k-1)d - k(k+1) + 1}$.*

Before we proceed to the proofs of Theorems 1 and 2 we collect some notation and terminology.

We consider finite, simple, and undirected graphs, and use standard terminology and notation. For a graph G, we denote the vertex set, edge set, order, and size by $V(G)$, $E(G)$, $n(G)$, and $m(G)$, respectively. If G has no cycle of length 3 or 5, then G is $\{C_3, C_5\}$-free. The square G^2 of a graph G has the same vertex set as G and two vertices are adjacent in G^2 if their distance in G is 1 or 2. Let $L(G)$ denote the line graph of G. For an edge e of G, let $C_G(e) = \{e\} \cup N_{L(G)^2}(e) = \{f \in E(G) : \text{dist}_{L(G)}(e, f) \leq 2\}$ and let $c_G(e) = |C_G(e)|$. Note that a set of edges of G is an induced matching if and only if it does not contain two distinct edges e and f with $f \in C_G(e)$, or, equivalently, $e \in C_G(f)$. In a maximal induced matching M of a graph G, for every edge f in $E(G) \setminus M$, there is some edge e in M with $f \in C_G(e)$. For some edges f, the choice of e might be unique, which motivates the following definition. For a set M of edges of G, let $PC_G(M, e) = C_G(e) \setminus \bigcup_{f \in M \setminus \{e\}} C_G(f)$ and let $pc_G(M, e) = |PC_G(M, e)|$. For two disjoint sets X and Y of vertices of G, let $E_G(X, Y)$ be the set of edges uv of G with $u \in X$ and $v \in Y$, and let $m_G(X, Y) = |E_G(X, Y)|$. For a set E of edges of G, let $G - E = (V(G), E(G) \setminus E)$. For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$.

2. Proofs

The greedy strategy for maximum induced matching relies on the following lemma.

Lemma 3. *Let G be a graph. If $G_0, \ldots, G_k$ are such that*

- *$G_0 = G$, and*
- *for $i \in [k]$, there is an edge e_i of G_{i-1} such that $G_i = G_{i-1} - C_{G_{i-1}}(e_i)$,*

then,

(i) *If e and f are edges of G_i for some $i \in [k]$, then $f \in C_{G_i}(e)$ if and only if $f \in C_G(e)$.*
(ii) *The set $\{e_1, \ldots, e_k\}$ is an induced matching.*

Proof. (i) Let $e = uv$ and $f = xy$ be edges of G_i for some $i \in [k]$. Since G_i is a subgraph of G, it follows that $f \in C_{G_i}(e)$ immediately implies $f \in C_G(e)$. Hence, for a contradiction, we may assume that $f \in C_G(e) \setminus C_{G_i}(e)$. Since $f \notin C_{G_i}(e)$, the two edges e and f do not share a vertex. Since $f \in C_G(e)$, we may assume that the graph G contains the edge ux. Since