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Linear-time algorithm for sliding tokens on trees

Erik D. Demaine [a], Martin L. Demaine [a], Eli Fox-Epstein [b], Duc A. Hoang [c,*], Takehiro Ito [d,e], Hirotaka Ono [f], Yota Otachi [c], Ryuhei Uehara [c], Takeshi Yamada [c]

[a] *MIT Computer Science and Artificial Intelligence Laboratory, 32 Vassar St., Cambridge, MA 02139, USA*
[b] *Department of Computer Science, Brown University, 115 Waterman Street, Providence, RI 02912-1910, USA*
[c] *School of Information Science, Japan Advanced Institute of Science and Technology, Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan*
[d] *Graduate School of Information Sciences, Tohoku University, Aoba-yama 6-6-05, Sendai, 980-8579, Japan*
[e] *CREST, JST, 4-1-8 Honcho, Kawaguchi, Saitama, 332-0012, Japan*
[f] *Faculty of Economics, Kyushu University, Hakozaki 6-19-1, Higashi-ku, Fukuoka, 812-8581, Japan*

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ABSTRACT

Suppose that we are given two independent sets I_b and I_r of a graph such that $|I_b| = |I_r|$, and imagine that a token is placed on each vertex in I_b. Then, the SLIDING TOKEN problem is to determine whether there exists a sequence of independent sets which transforms I_b into I_r so that each independent set in the sequence results from the previous one by sliding exactly one token along an edge in the graph. This problem is known to be PSPACE-complete even for planar graphs, and also for bounded treewidth graphs. In this paper, we thus study the problem restricted to trees, and give the following three results: (1) the decision problem is solvable in linear time; (2) for a yes-instance, we can find in quadratic time an actual sequence of independent sets between I_b and I_r whose length (i.e., the number of token-slides) is quadratic; and (3) there exists an infinite family of instances on paths for which any sequence requires quadratic length.

1. Introduction

Recently, *reconfiguration problems* have attracted the attention in the field of theoretical computer science. The problem arises when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible and each step conforms to a fixed reconfiguration rule (i.e., an adjacency relation defined on feasible solutions of the original problem). This kind of reconfiguration problem has been studied extensively for several well-known problems, including INDEPENDENT SET [2,5,7,11,12,14,16,20,22,23,25], SATISFIABILITY [10,21], SET COVER, CLIQUE, MATCHING [14], VERTEX-COLORING [3,6,8,25], LIST EDGE-COLORING [15,18], LIST $L(2,1)$-LABELING [17], SUBSET SUM [13], SHORTEST PATH [4,19], and so on. (See also a recent survey [24].)

1.1. SLIDING TOKEN

The SLIDING TOKEN problem was introduced by Hearn and Demaine [11] as a one-player game, which can be seen as a reconfiguration problem for INDEPENDENT SET. Recall that an *independent set* of a graph G is a vertex subset of G in which

* Corresponding author.
E-mail addresses: edemaine@mit.edu (E.D. Demaine), mdemaine@mit.edu (M.L. Demaine), ef@cs.brown.edu (E. Fox-Epstein), hoanganhduc@jaist.ac.jp (D.A. Hoang), takehiro@ecei.tohoku.ac.jp (T. Ito), hirotaka@econ.kyushu-u.ac.jp (H. Ono), otachi@jaist.ac.jp (Y. Otachi), uehara@jaist.ac.jp (R. Uehara), tyama@jaist.ac.jp (T. Yamada).

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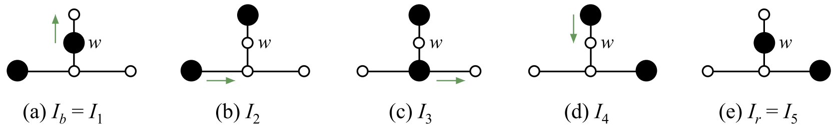

Fig. 1. A sequence $\langle I_1, I_2, \dots, I_5 \rangle$ of independent sets of the same graph, where the vertices in independent sets are depicted by large black circles (tokens).

Fig. 2. Two distinct independent sets I_b and I_r of the same star. This is a yes-instance for ISReconf under the TJ rule, but is a no-instance for the sliding token problem.

no two vertices are adjacent. (Fig. 1 depicts five different independent sets in the same graph.) Suppose that we are given two independent sets I_b and I_r of a graph $G = (V, E)$ such that $|I_b| = |I_r|$, and imagine that a token (coin) is placed on each vertex in I_b. Then, the sliding token problem is to determine whether there exists a sequence $\langle I_1, I_2, \dots, I_\ell \rangle$ of independent sets of G such that

(a) $I_1 = I_b$, $I_\ell = I_r$, and $|I_i| = |I_b| = |I_r|$ for all i, $1 \le i \le \ell$; and
(b) for each i, $2 \le i \le \ell$, there is an edge $\{u, v\}$ in G such that $I_{i-1} \setminus I_i = \{u\}$ and $I_i \setminus I_{i-1} = \{v\}$, that is, I_i can be obtained from I_{i-1} by sliding exactly one token on a vertex $u \in I_{i-1}$ to its adjacent vertex v along $\{u, v\} \in E$.

Such a sequence is called a *reconfiguration sequence* between I_b and I_r. Fig. 1 illustrates a reconfiguration sequence $\langle I_1, I_2, \dots, I_5 \rangle$ of independent sets which transforms $I_b = I_1$ into $I_r = I_5$. Hearn and Demaine proved that sliding token is PSPACE-complete for planar graphs, as an example of the application of their tool, called the nondeterministic constraint logic model, which can be used to prove PSPACE-hardness of many puzzles and games [11], [12, Sec. 9.5].

1.2. Related and known results

As the (ordinary) independent set problem is a key problem among thousands of NP-complete problems, sliding token plays an important role since several PSPACE-hardness results have been proved using reductions from it. In addition, reconfiguration problems for independent set (ISReconf, for short) have been studied under different reconfiguration rules, as follows.

- *Token Sliding* (TS rule) [6,7,11,12,20,25]: This rule corresponds to sliding token, that is, we can slide a single token only along an edge of a graph.
- *Token Jumping* (TJ rule) [7,16,20,25]: A single token can "jump" to any vertex (including a non-adjacent one) if it results in an independent set.
- *Token Addition and Removal* (TAR rule) [2,5,14,20,22,23,25]: We can either add or remove a single token at a time if it results in an independent set of cardinality at least a given threshold. Therefore, under the TAR rule, independent sets in the sequence do not have the same cardinality.

We note that the existence of a desired sequence depends deeply on the reconfiguration rules. (See Fig. 2 for example.) However, ISReconf is PSPACE-complete under any of the three reconfiguration rules for planar graphs [6,11,12], for perfect graphs [20], and for bounded bandwidth graphs [25]. The PSPACE-hardness implies that, unless NP = PSPACE, there exists an instance of sliding token which requires a super-polynomial number of token-slides even in a minimum-length reconfiguration sequence. In such a case, tokens should make "detours" to avoid violating independence. (For example, see the token placed on the vertex w in Fig. 1(a); it is moved twice even though $w \in I_b \cap I_r$.)

We here explain only the results which are strongly related to this paper, that is, sliding token on trees; see the references above for the other results.

1.2.1. Results for TS rule (sliding token)

Kamiński et al. [20] gave a linear-time algorithm to solve sliding token for cographs (also known as P_4-free graphs). They also showed that, for any yes-instance on cographs, two given independent sets I_b and I_r have a reconfiguration sequence such that no token makes a detour.

Very recently, Bonsma et al. [7] proved that sliding token can be solved in polynomial time for claw-free graphs. Note that neither cographs nor claw-free graphs contain trees as a (proper) subclass. Thus, the complexity status for trees was open under the TS rule.