Theoretical Computer Science 588 (2015) 16–23

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Any monotone property of 3-uniform hypergraphs is weakly evasive

Raghav Kulkarni [a], Youming Qiao [b,*], Xiaoming Sun [c]

[a] Centre for Quantum Technologies, the National University of Singapore, Singapore
[b] Centre for Quantum Computation and Intelligent Systems, University of Technology, Sydney, Australia
[c] CAS Key Lab of Network Data Science and Technology, Institute of Computing Technology, Chinese Academy of Sciences, 100190, Beijing, China

ARTICLE INFO

Article history:
Received 30 September 2013
Received in revised form 13 September 2014
Accepted 14 November 2014
Available online 26 November 2014

Keywords:
Decision tree complexity
Hypergraph properties
Evasiveness
Query complexity
Group actions

ABSTRACT

For a Boolean function f, let $D(f)$ denote its deterministic decision tree complexity, i.e., minimum number of (adaptive) queries required in worst case in order to determine f. In a classic paper, Rivest and Vuillemin [11] show that any non-constant monotone property $\mathcal{P}:\{0,1\}^{\binom{n}{2}} \to \{0,1\}$ of n-vertex graphs has $D(\mathcal{P}) = \Omega(n^2)$.
We extend their result to 3-uniform hypergraphs. In particular, we show that any non-constant monotone property $\mathcal{P}:\{0,1\}^{\binom{n}{3}} \to \{0,1\}$ of n-vertex 3-uniform hypergraphs has $D(\mathcal{P}) = \Omega(n^3)$.
Our proof combines the combinatorial approach of Rivest and Vuillemin with the topological approach of Kahn, Saks, and Sturtevant [6]. Interestingly, our proof makes use of Vinogradov's Theorem (weak Goldbach Conjecture), inspired by its recent use by Babai et al. [1] in the context of the topological approach. Our work leaves the generalization to k-uniform hypergraphs as an intriguing open question.

1. Introduction

The *decision tree model* aka *query model* [2], perhaps due to its simplicity and fundamental nature, has been extensively studied over decades; yet there remain some outstanding open questions about it.

Fix a Boolean function $f:\{0,1\}^n \to \{0,1\}$. A deterministic decision tree D_f for f takes $x=(x_1,\dots,x_n)$ as an input and determines the value of $f(x_1,\dots,x_n)$ using queries of the form "is $x_i = 1$?". Let $C(D_f, x)$ denote the cost of the computation, that is the number of queries made by D_f on input x. The *deterministic decision tree complexity* of f is defined as $D(f) = \min_{D_f} \max_x C(D_f, x)$.

The function f is called *evasive* if $D(f) = n$, i.e., one must query all the variables in worst case in order to determine the value of the function.

1.1. The Anderaa–Rosenberg–Karp Conjecture

A Boolean function f is said to be monotone (increasing) if for any $x \le y$ we have $f(x) \le f(y)$, where $x \le y$ iff for all $i : x_i \le y_i$. A property of n-vertex graphs is a Boolean function $\mathcal{P}:\{0,1\}^{\binom{n}{2}} \to \{0,1\}$ whose variables are identified with the

* Corresponding author.
E-mail addresses: kulraghav@gmail.com (R. Kulkarni), jimmyqiao86@gmail.com (Y. Qiao), sunxiaoming@ict.ac.cn (X. Sun).

http://dx.doi.org/10.1016/j.tcs.2014.11.012
0304-3975/

$\binom{n}{2}$ potential edges of n-vertex graphs and the function $\mathcal{P}$ is invariant under relabeling of the vertices. $\mathcal{P}(G) = 0$ means that the graph G satisfies the property. A natural theme in the study of decision tree complexity is to exploit the structure within f to prove strong lower bounds on its query complexity. A classic example is the following conjecture attributed to Anderaa, Rosenberg, and Karp, asserting the *evasiveness* of monotone graph properties:

Conjecture 1.1 *(ARK Conjecture). (Cf. [6].) Every non-trivial monotone graph property is evasive.*

Some natural examples of monotone graph properties are: connectedness, planarity, 3-colorability, containment of a fixed subgraph etc.

Since its origin around 1975, the ARK Conjecture has caught the imagination of generations of researchers resulting in beautiful mathematical ideas; yet – to this date – it remains unsolved. A major breakthrough on ARK Conjecture was obtained by Kahn, Saks, and Sturtevant [6] via their novel *topological approach*. They settled the conjecture when the number of vertices of the graphs is a power of prime number. The topological approach subsequently turned out useful for solving some other variants and special cases of the conjecture. For example: Yao confirms the variant of the conjecture for monotone properties of bipartite graphs [14]. More recently, building on Chakrabarti, Khot, and Shi's work [3], Babai et al. [1] show that under some well-known conjectures in number theory, *forbidden subgraph* property – containment of a fixed subgraph in the graph – is evasive. We refer the readers to the lecture notes [7] by Lovász and Young for a nice exposition of the works around this topic.

1.2. The evasiveness conjecture

The key feature of monotone graph properties is that they are sufficiently *symmetric*. In particular, they are *transitive* Boolean functions, i.e., there is a group acting transitively on the set of variables under which the function remains invariant. A natural question was raised: how much *symmetry* is necessary in order to guarantee the evasiveness? The following generalization (cf. [8]) of ARK Conjecture asserts that only transitivity suffices.

Conjecture 1.2 *(Evasiveness Conjecture (EC)). If f is a non-trivial monotone transitive Boolean function, then f is evasive.*

Rivest and Vuillemin [11] confirm the above conjecture when the number of variables is a power of prime number. The general case remains widely open.

1.3. The weak evasiveness conjecture

Recently Kulkarni [5] proposed to investigate the following:

Conjecture 1.3 *(Weak Evasiveness Conjecture). If $\{f_n\}$ is a sequence of non-trivial monotone transitive Boolean functions then for every $\epsilon > 0$*

$$D(f_n) \geq n^{1-\epsilon}.$$

The best known lower bound in this context is $D(f) \geq R(f) \geq n^{2/3}$, which follows from the work of O'Donnell et al. [10]. It turns out that [5] the above conjecture is equivalent to the EC! Furthermore: the Rivest and Vuillemin [11] result, which settles the ARK Conjecture up to a constant factor, in fact confirms the Weak-EC for graph properties.

Theorem 1.4 *(Rivest and Vuillemin). If $\mathcal{P} : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ is a non-trivial monotone property of graphs on n vertices then $D(\mathcal{P}) = \Omega(n^2)$.*

It is interesting to note that the proof of equivalence in Kulkarni [5] does not hold between ARK and Weak-ARK. Hence: even though Weak-ARK is settled, the ARK is still wide open.

1.4. Our results on the weak EC

In this paper we prove an analogue of Rivest and Vuillemin's result (Theorem 1.4) for 3-uniform hypergraphs. A property of 3-uniform hypergraphs on n vertices is a Boolean function $\mathcal{P} : \{0,1\}^{\binom{n}{3}} \to \{0,1\}$ whose variables are labeled by the $\binom{n}{3}$ potential edges of n-vertex 3-uniform hypergraphs and $\mathcal{P}$ is invariant under relabeling of the vertices.

Theorem 1.5. *If $\mathcal{P} : \{0,1\}^{\binom{n}{3}} \to \{0,1\}$ is a non-trivial monotone property of 3-uniform hypergraphs on n vertices, then*

$$D(\mathcal{P}) = \Omega\left(n^3\right).$$