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On maximum independent set of categorical product and ultimate categorical ratios of graphs

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ABSTRACT

We first present polynomial algorithms to compute the independence number of the categorical product for two cographs or two splitgraphs, respectively. Then we prove that computing the maximum independent set of the categorical product of a planar graph of maximum degree three and a K_4 is NP-complete. The ultimate categorical independence ratio of a graph G is defined as $\lim_{k\to\infty} \alpha(G^k)/n^k$. The ultimate categorical independence ratio can be computed in polynomial time for cographs, splitgraphs, permutation graphs, interval graphs and graphs of bounded treewidth. Also, we present an $O^*(3^{n/3})$-time exact, exponential algorithm for the ultimate categorical independence ratio of general graphs. We further present a PTAS for the ultimate categorical independence ratio of planar graphs. Lastly, we show that the ultimate categorical independent domination ratio for complete multipartite graphs is zero, except when the graph is complete bipartite with color classes of equal size (in which case it is 1/2).

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1. Introduction

Let G and H be two graphs. The categorical product also travels under the guise of tensor product, or direct product, or Kronecker product, and even more names have been given to it. It is defined as follows. It is a graph, denoted by $G \times H$. Its vertices are the ordered pairs (g, h) where $g \in V(G)$ and $h \in V(H)$. Two of its vertices, say (g_1, h_1) and (g_2, h_2) are adjacent if and only if

$$\{g_1, g_2\} \in E(G) \quad \text{and} \quad \{h_1, h_2\} \in E(H).$$

One of the reasons for its popularity is Hedetniemi's conjecture, which is now more than 40 years old [18,37,39,47].

Conjecture 1. *(See [18].) For any two graphs G and H,*

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$$\boxed{\textit{Hedetniemi's conjecture:}\quad \chi(G \times H) = \min\{\, \chi(G),\ \chi(H)\,\},}$$

where $\chi(I)$ denotes the chromatic number of a graph I. It is easy to see that the right-hand side is an upperbound. Namely, if f is a vertex coloring of G, then one can color $G \times H$ by defining a coloring f' as follows

$$f'((g,h)) = f(g), \quad \textit{for all } g \in V(G) \textit{ and } h \in V(H).$$

Recently, Zhu [48] showed that the fractional version of Hedetniemi's conjecture is true.

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required to color the vertices of G so that the adjacent vertices do not have the same color. The *clique number* $\omega(H)$ of a graph G is the size of a maximum clique in G. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. When restricted to perfect graphs, say G and H, Hedetniemi's conjecture is true. Namely, let K be a clique of cardinality

$$|K| \le \min\{\, \omega(G),\ \omega(H)\,\}.$$

It is easy to check that $G \times H$ has a clique of cardinality $|K|$. One obtains an 'elegant' proof via homomorphisms as follows. By assumption, there exist homomorphisms $K \to G$ and $K \to H$. This implies that there is also a homomorphism $K \to G \times H$ (see, e.g., [16,19]). (Actually, if W, P and Q are any graphs, then there exist homomorphisms $W \to P$ and $W \to Q$ if and only if there exists a homomorphism $W \to P \times Q$.) In other words [16, Observation 5.1],

$$\omega(G \times H) \ge \min\{\, \omega(G),\ \omega(H)\,\}.$$

Since G and H are perfect, $\omega(G) = \chi(G)$ and $\omega(H) = \chi(H)$. This proves the claim, since

$$\begin{aligned} \chi(G \times H) \ge \omega(G \times H) &\ge \min\{\, \omega(G),\ \omega(H)\,\} \\ &= \min\{\, \chi(G),\ \chi(H)\,\} \ge \chi(G \times H). \end{aligned} \tag{1}$$

Since much less is known about the independence number $\alpha(G \times H)$ of the categorical products of two graphs G and H, we are motivated to study this problem. It is easy to see that

$$\alpha(G \times H) \ge \max\{\, \alpha(G) \cdot |V(H)|,\ \alpha(H) \cdot |V(G)|\,\}. \tag{2}$$

But this lowerbound can be arbitrarily bad, even for threshold graphs [21]. For any graph G and any natural number k, there exists a threshold graph H such that

$$\alpha(G \times H) \ge k + L(G,H),$$

where $L(G,H)$ is the lowerbound shown in Eq. (2). Zhang [46] recently proved that, when G and H are vertex transitive, then equality holds in Eq. (2). We consider the computation of the independence number of the categorical product $G \times H$ for cographs, splitgraphs, or other graph classes, respectively. The formal definitions of cographs and splitgraphs are as follows.

Definition 1. A graph is a *cograph* if it contains no induced P_4, which is a path with four vertices.

More details about this class of graphs will be described in Section 2.2.

Földes and Hammer [12] introduced splitgraphs.

Definition 2. (See [12].) A graph G is a *splitgraph* if there is a partition $\{S, C\}$ of its vertices such that $G[C]$ is a clique and $G[S]$ is an independent set.

We refer to [13, Chapter 6] and [29] for some background information on this class of graphs.

We then proceed to consider a more general product, the *k-fold categorical product* $G^k = G \times \ldots \times G$ of k copies of G for $k \to \infty$. Notice that, when G is vertex transitive then G^k is also vertex transitive and so, by the "no-homomorphism" lemma of Albertson and Collins [1], $\alpha(G^k) = \alpha(G) \cdot n^{k-1}$.

Since the independence number $\alpha(G^k)$ may not converge when $k \to \infty$, the target is, instead, to compute the ratio of the independence number $\alpha(G^k)$ versus the number of vertices of G^k. The formal definition of the ratio is given as follows.

Definition 3. The *independence ratio* of a graph G is defined as

$$r(G) = \frac{\alpha(G)}{|V(G)|}. \tag{3}$$

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