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The Time Invariance Principle, the absence of ecological chaos, and a fundamental pitfall of discrete modeling

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ABSTRACT

This paper is to show that most discrete models used for population dynamics in ecology are inherently pathological that their predications cannot be independently verified by experiments because they violate a fundamental principle of physics. The result is used to tackle an on-going controversy regarding ecological chaos. Another implication of the result is that all dynamical systems must be modeled by differential equations. As a result it suggests that researches based on discrete modeling must be closely scrutinized and the teaching of calculus and differential equations must be emphasized for students of biology.

1. Introduction

No models in ecology are better known than the Logistic Map, or have played a greater role in the development of the chaos theory (May, 1974; Hassel, 1975; Hassel et al., 1975; Berryman and Millstein, 1989; Logan and Allen, 1992). Surprisingly, however, there is not a greater controversy than what was generated by the model's prediction that one-species populations are inherently chaotic.

The key prediction of the Logistic Map, $x_{n+1} = Q(x_n, r) := r x_n (1 - x_n)$, says that increasing the intrinsic reproductive rate r leads to chaotic oscillations in population. However, contradicting evidence existed even before the chaos theory was popularized in ecology. One noticeable example was given by McAllister and LeBrasseur (1971) who showed that enriching an aquatic system led to stable equilibrium. Ensuing extensive search for field chaos came up empty-handed. For example, well-established geographic patterns on microtine species (Hanski et al., 1991; Falck et al., 1995) showed that ecological systems tend to stabilize down the north-to-south latitude gradient, correlating well with the ultimate resource abundance in liquid water and sunlight towards the equator. A comprehensive hunt for ecological chaos was down by Ellner and Turchin (1995) who used three

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different Lyapunove exponent estimators to analyze a large collection of empirical data and showed rather conclusively that ecological chaos is not to be expected in the wild.

The glaring irreconcilability between the theory and reality can only lead to one logical conclusion: the theory is wrong. Concluding it otherwise would have to imply that logic imperatives do not apply to ecology. However, a definitive explanation to the theory's failure is lacking while efforts to justify it continue (e.g. Eskola and Geritz, 2007). The purpose of this paper is to make a case that the Logistic Map and most discrete maps used in ecology and life sciences cannot be models for *any* physical process, population dynamics in particular because their predictions cannot be *independently* verified by experiments.

2. The result

This conclusion rests on a fundamental principle of physics held since the time of Copernicus in the 15th century that a physical law should be the same anywhere and anytime in the universe. In other words, a law must take the same mathematical form, derivable from experiments carried out at independently chosen times and spaces. As a result, the mathematical formulation of a law must be endowed with such time invariance property. Taken to be self-evident, we state the principle in the following formulation more suited for the issues under consideration:

> *Time Invariance Principle* (*TIP*): A physical law has the same mathematical form to every independent choice of observation time.

This principle has an important implication to dynamical systems as laws of physical processes. To be precise, let y be the set of state variables and p be the set of parameters of a physical process. As a dynamical system, y changes in time t. Suppose an observation is made at t_0 and the state is y_0, where t_0 is the time passage since the start of the process. Another observation is made $t>0$ time after t_0 and the state is y_t. Then, as a physical law, y_t is governed by a function, denoted by $y_t=\psi_t(y_0, t_0, p)$, depending on the passage of time t beyond t_0, the state y_0 at t_0, and the system parameter p. As a default requirement, it must satisfy the unitary condition

$$\psi_0(y_0, t_0, p) = y_0,$$

that is, with time increment 0, the law ψ_0 leaves every state fixed. Now by the Time Invariance Principle, if another observation is made $s>0$ unit time later, the same functional form $(y_t)_s=\psi_s(y_t, t+t_0, p)$ must hold. Most importantly, the function ψ_t must satisfy the following group property and the unitary condition

$$(y_t)_s = \psi_s(\psi_t(y_0, t_0, p), t+t_0, p) = \psi_{s+t}(y_0, t_0, p) = y_{s+t},$$
$$\text{and}\quad \psi_0(y_0, t_0, p) = y_0, \tag{1}$$

which together is referred to being *TIP-conforming*. That is, if an observation is made t time after the initial observation, and another is made s time later, then the result must be the same if only one observation is made $s+t$ time after the initial observation. More generally, the state at $s+t$ after the state y_0 at t_0 is the same state at s after an intermediate state y_t which is the state at t after the same initial y_0 at t_0. A violation of this property that $\psi_{s+t}(y_0, t_0, p)\neq\psi_s(\psi_t(y_0, t_0, p), t+t_0, p)$ implies that either such an "experiment" is not reproducible, i.e., using independent observing times lead to irreconcilable conclusions, or such a functional form ψ does not govern the laws that the experiment is to establish or to verify.

A physical process is called *autonomous* if its dynamical law $\psi_t(y_0, t_0, p)$ is independent of t_0. In fact, every process can be considered as autonomous by augmenting the state only one-dimension higher. More specifically, let $x=(y, \tau)$ and denote $x=(y_0, \tau_0)=(y_0, t_0)$, $x_t=(y_t, \tau_t)$ with

$$\tau_t = t + \tau_0 = t + t_0,$$

Then the augmented state x is autonomous even if y is not. More specifically, let

$$\phi_t(x_0, p) = (\psi_t(y_0, \tau_0, p), t + \tau_0),$$

then it is straightforward to check the following

Lemma 1. *The functional form ψ satisfies the TIP-conforming property (1) if and only if the augmented functional form ϕ satisfies the autonomous TIP-conforming property*

$$(x_t)_s = \phi_s(x_t, p) = \phi_s(\phi_t(x_0, p), p) = \phi_{s+t}(x_0, p),$$
$$\text{and}\quad \phi_0(x_0, p) = x_0, \tag{2}$$

Thus, from now on we will assume all TIP-conforming functional forms are autonomous, and both properties (1) and (2) are interchangeably referred to as the *TIP-conforming group property*.

As a result, an immediate consequence to the Time Invariance Principle is the following.

Lemma 2. *If a TIP-conforming dynamical system $\phi_t(x, p)$ is continuously differentiable at $t=0$ and any x in its domain of definition, then $x(t)=\phi_t(x_0, p)$ must be the unique solution to an initial value problem of a differential equation:*

$$\frac{dx(t)}{dt} = F_\phi(x(t), p), \quad x(0) = x_0,$$

where

$$F_\phi(x, p) = \frac{\partial\phi_h}{\partial h}(x, p)|_{h=0}$$

is the generating vector field of ϕ_t. Conversely, if the vector field F is continuous differentiable, then the solution to the initial value problem satisfies the TIP-conforming group property (2).

Proof. Because ϕ is differentiable and is TIP-conforming (2), we have the following derivative

$$\begin{aligned}\frac{dx(t)}{dt} &= \lim_{h\to 0}\frac{\phi_{t+h}(x_0, p) - \phi_t(x_0, p)}{h}\\ &= \lim_{h\to 0}\frac{\phi_h(\phi_t(x_0, p), p) - \phi_0(\phi_t(x_0, p), p)}{h}\\ &= \frac{\partial\phi_h}{\partial h}(\phi_t(x_0, p), p)|_{h=0} = F_\phi(x(t), p),\end{aligned}$$