Theoretical Computer Science 583 (2015) 27–39

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Computability aspects for 1st-order partial differential equations via characteristics

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ARTICLE INFO

Article history:
Received 11 October 2013
Received in revised form 16 March 2015
Accepted 21 March 2015
Available online 31 March 2015
Communicated by O. Watanabe

Keywords:
Computability analysis
1st-order PDEs
Method of characteristics

ABSTRACT

In this paper, it is shown that the method of characteristics can be used to compute local solutions of the boundary value problems for the first-order partial differential equations at feasible instances, under the natural definition of computability from the view point of application; but the maximal region of existence of a computable local solution may not be computable. It is also shown that the problem whether a boundary value problem has a global solution is not algorithmically decidable. The negative results retain even within the class of quasilinear equations defined by analytic computable functions over particularly simple domains (quasilinear equations are among the simplest first-order nonlinear partial differential equations). This fact shows that the algorithmic unsolvability is intrinsic.

1. Introduction

Computable analysis of differential equations studies the equations whose solutions can be computed with arbitrary precision on machines such as digital computers.

Computable analysis of ordinary differential equations (ODEs) can be dated back to 1970s (see, for example, Aberth [1], Pour-El/Ricards [12]); many results on computability and complexity of solutions to the initial value problems (IVPs) for ODEs have been obtained since then (see, for example, [1,12,9,4,13,8] and the references therein). In contrast, there are only a few theoretic studies on computability of partial differential equations (PDEs); most of them concern particular equations, such as the KdV equation [17,15,3] and the Schrödinger equations [16].

In this paper, we study computability aspects for a class of partial differential equations, namely, general first-order PDEs for scalar functions with several independent variables. Equations of this type occur naturally in the calculus of variations, in particle mechanics, and in geometrical optics. Although the nonlinearity of such equations often precludes our attempts of deriving explicit solution formulas, we may still apply simple calculus to obtain fair amount of detailed information on the solutions. In particular, for general first-order partial differential equations, a technique called the method of characteristics plays an important role. The main idea of this method is to reduce the boundary value problem (BVP) for a PDE to a family of initial value problems for certain ODEs along some boundary curves; once the ODEs are solved, their solutions are then transformed into a solution for the original PDE problem. Since the theory of computability for the first-order ODEs is well established, the method of characteristics prompts a natural question: What is the connection between the computability of

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1 The research was partially supported by the National Science Foundation under grant No. DMS-1210979.

http://dx.doi.org/10.1016/j.tcs.2015.03.039
0304-3975/

the solution of the original PDE and the solutions of the ODEs which result from it? In this paper, we explore the answers to this question.

The paper is organized as follows. Section 2 introduces necessary definitions and notations from the theory of the BVPs for the first-order PDEs, computability theory and computable analysis. Section 3 presents a theorem stating that the solution of a BVP is computable near a feasible instance, provided that the data defining the BVP are computable. Section 4 discusses computability of boundary conditions. Section 5 contains several counterexamples showing that the problem of determining the maximal region of existence of a local solution and the problem of existence of global solutions are not algorithmically solvable, even within a class of particularly simple equations defined on especially nice domains.

2. Notations and definitions

We investigate the computability of solutions to a general nonlinear first-order partial differential equation of the form

$$F(Du, u, x) = 0 \quad \text{in } U \tag{1}$$

subject to the boundary condition

$$u = g \quad \text{on } \partial U \text{ (the boundary of } U) \tag{2}$$

where $x \in U$, U is an open subset of $\mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R} \times V \to \mathbb{R}$ and $g : \partial U \to \mathbb{R}$ are given, V is an open subset of $\mathbb{R}^n$ containing $\overline{U}$, $u : \overline{U} \to \mathbb{R}$ is the unknown, $u = u(x)$, $Du = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n})$ is the gradient, and $\overline{U}$ denotes the closure of U in $\mathbb{R}^n$. We call (1), (2) a boundary value problem or a BVP in short. A function u is called a (strong) solution of the boundary value problem if it solves (1), (2).

Remark. It can be assumed that g is the restriction of a function G defined in a neighborhood of ∂U. Classically it is known that any C^k function g defined on ∂U can be extended to a C^k function G defined in a neighborhood of ∂U, provided that ∂U is C^{k+1} smooth (the definition of a C^k smooth ∂U is given in the paragraph before Definition 1). We will make use of this convention in this paper.

The plan is to convert, using the method of characteristics, the boundary value problem (1), (2) into an appropriate initial value problem for some first-order ODEs. The idea underlying the method of characteristics can be highlighted as follows: to compute the solution $u(x)$ at a point $x \in U$, one finds some curve $\mathbf{x}(s)$ lying within U, connecting x with a point $x^0 \in \partial U$, such that along the curve $\mathbf{x}(s)$ the boundary value problem for the PDE is reduced to an initial value problem for a system of first-order ODEs (called the characteristic equations of the PDE (1)), starting at the one end x^0 of the curve $\mathbf{x}(s)$ with $u(x^0) = g(x^0)$, which is given.

We begin by recalling some definitions and notations. In the following, we use superscripts for points or for component functions of a vector-valued function and subscripts for coordinates of points. Let us write

$$F = F(p, z, x) = F(p_1, p_2, \dots, p_n, z, x_1, x_2, \dots, x_n)$$

and

$$D_p F = (F_{p_1}, F_{p_2}, \dots, F_{p_n})$$
$$D_z F = F_z$$
$$D_x F = (F_{x_1}, F_{x_2}, \dots, F_{x_n})$$

where $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x = (x_1, x_2, \dots, x_n) \in V$, and F_w denotes the partial derivative of F with respect to the variable w.

Classically, it is known that the following system of $2n + 1$ first-order ODEs comprises the characteristic equations of the nonlinear PDE (1):

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) & \text{(a)} \\ \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) & \text{(b)} \\ \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) & \text{(c)} \end{cases} \tag{3}$$

where $\dot{} = \frac{d}{ds}$ and $\cdot$ denotes the dot product. The bold-faced $\mathbf{p}$ and $\mathbf{x}$ are used for the vector-valued functions; in particular, $\mathbf{p}$ is a map from some interval $I \subseteq \mathbb{R}$ to $\mathbb{R}^n$ with $\mathbf{p}(s) = (p^1(s), \dots, p^n(s))$ and $\mathbf{x}(s)$ denotes a parametric curve with the parameter s lying in I, i.e. $\mathbf{x} : I \to \mathbb{R}^n$, $\mathbf{x}(s) = (x^1(s), \dots, x^n(s))$.

Note that the PDE (1) may have no solution or may have many solutions. Nevertheless, one can verify that if $u \in C^2(U)$ is a solution to the PDE (1) and $\mathbf{x}(s)$ solves the ODE (c), where $\mathbf{p}(s) = Du(\mathbf{x}(s))$ and $z(s) = u(\mathbf{x}(s))$, then $\mathbf{p}(s)$ solves the ODE (a) and $z(s)$ solves the ODE (b), for those values of s such that $\mathbf{x}(s) \in U$ (cf. Theorem 1 of Section 3.2 in [2]). The implication of this fact is that the characteristic (3) are necessary conditions for the solutions of the PDE (1), but may not be sufficient.