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Feasibility checking in Horn constraint systems through a reduction based approach ☆

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ABSTRACT

In this paper, we detail a new algorithm for the problem of checking linear and integer feasibility in a system of Horn constraints. For certain special cases, the proposed algorithm is faster than the "Lifting Algorithm" for Horn constraint feasibility described in [2]. Moreover, the new approach is based on different ideas and in fact exploits several properties of Horn constraint systems (HCS) which were heretofore unknown, to the best of our knowledge. In the case of constraints of bounded width (corresponding to "loosely coupled" systems), our algorithm can be modified to run in $O(n^3 + m \cdot n + \frac{m \cdot n^2}{\log(\max(m,n))})$ time, where n and m represent the number of variables and the number of constraints respectively, in the input HCS. Our main result establishes that checking the feasibility of an HCS can be reduced to three subproblems: negative-cost cycle detection in networks (NCCD), matrix–vector multiplication (MV), and the conversion of an HCS to a non-redundant set of difference constraints (H2D). The MV problem and the NCCD problem have both been studied extensively, as per the literature and there exist specialized, fast algorithms for the cases that are relevant to feasibility checking in Horn constraint systems. We have identified a new problem, viz., H2D, which warrants future research, since improved algorithms for the H2D problem could be implemented in our algorithm to decrease its running time even further.

1. Introduction

In this paper, we introduce a new algorithm for the problem of checking the feasibility of a conjunction of linear Horn constraints. Horn constraint systems (HCS) generalize Difference constraint systems (DCS) and find applications in a number of domains, including but not limited to program verification and econometrics. This work builds upon our work in [2], wherein the first combinatorial algorithm for this problem was proposed. The algorithm that we propose in this paper, is based on substantially different ideas than those in [2], and in fact exploits a number of properties of Horn constraint systems which are not known to be part of the literature. Essentially, we Turing-reduce the Horn constraint feasibility problem to three subproblems, viz., negative-cost cycle detection (NCCD), matrix–vector multiplication (MV), and the conversion of a system of Horn constraints to a non-redundant system of difference constraints (H2D). The new algorithm may be either combinatorial or non-combinatorial, depending on the algorithms chosen for the NCCD, MV, and H2D problems. Furthermore, the complexity of our algorithm is expressed in terms of the complexities of the NCCD, MV, and H2D problems.

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Accordingly, an improvement in the running times of algorithms for these three problems leads to an immediate improvement in the running time of the proposed algorithm. In the case of *bounded width* constraints (see Section 2), the algorithm can be modified to run in time $O(n^3 + m \cdot n + \frac{m \cdot n^2}{\log(\max(m,n))})$, where n and m represent the number of variables and number of constraints respectively, in the input HCS. To the best of our knowledge, our algorithm is the fastest known in this case.

The main contributions of this paper are as follows:

(i) Design and analysis of a new algorithm for checking feasibility in Horn constraint systems, and
(ii) Extending the analysis to handle the case of Extended Horn constraints.

The rest of this paper is organized as follows: Section 2 formally specifies the problem under consideration. A brief description of the Lifting Algorithm for Horn constraints is provided in Section 3. This section also includes a lemma which is essential for proving the correctness of our algorithm. In Section 4, we discuss the motivation for our work. Related approaches in the literature are detailed in Section 5. Section 6 describes a technique by which an HCS can be converted into a difference constraint system (DCS), such that the infeasibility of the DCS guarantees the infeasibility of the HCS. Section 7 presents the new algorithm for checking feasibility in an HCS. Section 8 examines the complexity of the proposed algorithm. The techniques detailed in Section 7, are used in Section 9, to develop a new algorithm for a larger class of constraints called Extended Horn constraints. We conclude in Section 10 by summarizing our contributions and outlining avenues for future research. In order to simplify the exposition of the proof of correctness of the feasibility algorithm, we only consider *standardized* Horn constraint systems. A procedure for converting an arbitrary HCS into a standardized one is described in Appendix A.

2. Statement of problem

Let

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x} &\geq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned} \tag{1}$$

denote a *polyhedral system* in which $\mathbf{A}$ is an $m \times n$ integral matrix, $\mathbf{b}$ is an integral m-vector, and $\mathbf{x} = [x_1, x_2, \dots, x_n]^{\mathrm{T}}$ is a rational n-vector. For notational convenience, the sub-system $\mathbf{A} \cdot \mathbf{x} \geq \mathbf{b}$ is referred to as the defining constraint system and the constraints $\mathbf{x} \geq \mathbf{0}$ are referred to as the non-negativity constraints. We note that the non-negativity constraints could be included in the defining constraint system; however, the exposition of our algorithm is simplified, when they are treated as separate identities.

Definition 1. A polyhedral system is said to be a *Difference Constraint System* (DCS) if each row of $\mathbf{A}$ contains at most two non-zero entries with one of these entries (if any) being 1 and the other (if any) being -1.

For instance, $x_1 - x_2 \geq 3$ is a difference constraint.

Definition 2. A polyhedral system is said to be a *Horn Constraint System* (HCS) or a *Horn polyhedron* if

(i) the entries of $\mathbf{A}$ belong to the set $\{0, 1, -1\}$, and,
(ii) each row of $\mathbf{A}$ contains at most one positive entry.

The matrix $\mathbf{A}$ is said to satisfy the Horn structure.

For instance, $x_1 - x_5 - x_7 \geq -3$ is a Horn constraint. An HCS could include *absolute* constraints (also called *unary* constraints), i.e., constraints of the form $x_1 \geq 5$ or $-x_2 \geq -6$. Note that a constraint such as $x_1 \geq 5$ can be replaced by the constraint $x_1' \geq 0$, where $x_1' = x_1 - 5$, and the resultant constraint system is feasible if and only if the original system is. Note also that (as we discuss in Section 7) constraints of the form $-x_2 \geq -6$ will be ignored until after the execution of our proposed algorithm. A special case of HCS is defined next.

Definition 3. An HCS is said to be of *bounded width k*, if every constraint contains at most k non-zero entries.

For example, a DCS is an HCS of bounded width 2.

The definitions presented next extend Definitions 1 and 2 respectively.

Definition 4. A polyhedral system is said to be an *Extended Difference Constraint System* (EDCS) if each row of $\mathbf{A}$ contains at most two non-zero entries with one of these entries (if any) being 1 and the other (if any) being an arbitrary negative integer.