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Matching preclusion and conditional matching preclusion problems for the folded Petersen cube

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ABSTRACT

The matching preclusion number of an even graph is the minimum number of edges whose deletion results in a graph that has no perfect matchings. For many interconnection networks, the optimal sets are precisely those induced by a single vertex. Recently, the conditional matching preclusion number of an even graph was introduced to look for obstruction sets beyond those induced by a single vertex. It is defined to be the minimum number of edges whose deletion results in a graph *with no isolated vertices* and no perfect matchings. In this paper, we study this problem for the folded Petersen cube $FPQ(n, k)$ via some matching preclusion and conditional matching preclusion results of the Cartesian products of graphs.

1. Introduction

A *perfect matching* in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. Throughout the paper, we only consider *even graphs*, that is, graphs with an even number of vertices. A set of edges F is called a *matching preclusion set* if $G - F$ has no perfect matching, and it is called an *optimal matching preclusion set* if F is one with the smallest size. We refer to the elements in F *faulty* or call them the *fault-edges*. If F is not a matching preclusion set, that is, $G - F$ contains a perfect matching, we refer to such a matching *fault-free perfect matching with respect to* F or simply *fault-free perfect matching* if it is clear from the context. The *matching preclusion number* of an even graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching. So $\mathrm{mp}(G) = 0$ if G has no perfect matchings. This concept of matching preclusion was introduced by [1] and further studied by [2,5,6,10,12,18,19]. They introduced this concept as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. In interconnection networks, it is desirable to have the property that all optimal matching preclusion sets are those whose elements are incident to a single vertex. We refer the readers to [1] for details and additional references.

Proposition 1.1. *Let G be an even graph. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proof. Deleting all edges incident to a single vertex of minimum degree will give a graph with no perfect matchings and the result follows. $\square$

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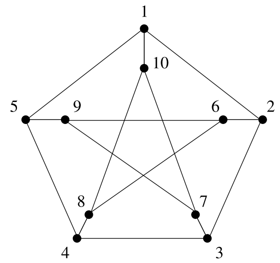

Fig. 1. The Petersen graph.

We call an optimal solution of the form given in the proof of Proposition 1.1 a *trivial optimal matching preclusion set*. As mentioned earlier, it is desirable for an interconnection network to have only trivial optimal matching preclusion sets. If $\mathrm{mp}(G) = \delta(G)$, then G is *maximally matched*. If, in addition, every optimal matching preclusion set is a trivial optimal matching preclusion set, then G is *super matched*. Given that it is unlikely that in the event of random link failure, the failed edges contain all edges incident to some vertex, it is natural to ask what are the next obstruction sets for a graph with link failures to have a perfect matching subject to the condition that the *faulty graph* has no isolated vertices. This motivates the definition given in [3] and further studied in [4,13,18]. A *conditional matching preclusion set* is a set of edges whose deletion leaves the resulting graph with no isolated vertices and no perfect matchings. We call a conditional matching preclusion set of smallest size an *optimal conditional matching preclusion set*. The *conditional matching preclusion number* of a graph G, denoted by $\mathrm{mp}_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and no perfect matchings. We will leave $\mathrm{mp}_1(G)$ undefined if a conditional matching preclusion set does not exist, that is, we cannot delete edges to satisfy both conditions in the definition.

Now for a resulting graph with no isolated vertices, a basic obstruction to a perfect matching will be the existence of a path u–w–v where the degree of u and the degree of v are 1. So to produce such an obstruction set, one can pick any path u–w–v in the original graph and delete all the edges incident to either u or v but not the edges (u, w) and (w, v). We define $\nu_e(G)$ to be

$$\min\{d_G(u) + d_G(v) - 2 - y_G(u, v) \;:\; u \text{ and } v \text{ are ends of a 2-path}\}^{1}$$

where $d_G(\cdot)$ is the degree function and $y_G(u, v) = 1$ if u and v are adjacent and 0 otherwise. (We will suppress G and simply write d and y if it is clear from the context.) So mirroring Proposition 1.1, we have the following easy result.

Proposition 1.2. *Let G be an even graph with $\delta(G) \geq 3$. Then*

$$mp_1(G) \leq \nu_e(G).$$

We call an optimal solution of the form induced by ν_e a *trivial optimal conditional matching preclusion set*. For interconnection networks, it is desirable to have the property that all optimal conditional matching preclusion sets are trivial. If $\mathrm{mp}_1(G) = \nu_e(G)$, then G is *conditionally maximally matched*. If, in addition, every optimal conditional matching preclusion set is trivial, then G is *conditionally super matched*. These concepts were introduced [3] and this problem was also studied by [4,6,13,18].

The class of folded Petersen cubes was introduced as a competitive model of the hypercubes. Some recent research includes [9,14]. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. Then their *Cartesian product* $G \square H$ is the graph with vertex set $V_G \times V_H = \{(g, h) : g \in V_G, h \in V_H\}$, and the edges (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and $(h_1, h_2) \in E_H$, or $(g_1, g_2) \in E_G$ and $h_1 = h_2$. It is easy to see that $\square$ is associative and commutative under isomorphism. The *hypercube* Q_n (where $n \geq 1$) is usually defined by taking the set of $\{0, 1\}$-strings of length n as the vertex set, and two vertices are adjacent if and only if they differ in exactly one position. Then it is easy to see that Q_n is isomorphic to K_2^n, that is, $K_2 \square K_2 \square \cdots \square K_2$ with n K_2's, where K_2 is the complete graph on two vertices. Let P be the Petersen graph. See Fig. 1. Then $FP_k = P^k$ (where $k \geq 1$) is the k-dimensional *folded Petersen graph*, $HP_n = P \square Q_{n-3}$ (where $n \geq 3$) is the *hyper Petersen graph* [8]. The *folded Petersen cube* is the graph $FPQ(n, k) = P^k \square Q_n = P^k \square K_2^n$ where $n \geq 0$, $k \geq 0$ and $(n, k) \neq (0, 0)$. So $FPQ(n, k)$ is a triangle free $(n + 3k)$-regular graph with $10^k 2^n$ vertices. We note that since $FPQ(n, k)$ is triangle-free, $\nu_e(FPQ(n, k)) = (n + 3k - 1) + (n + 3k - 1) = 2n + 6k - 2$. Thus the class of folded Petersen cubes include the hypercubes, folded Petersen graphs and hyper Petersen graphs. It was shown that a hyper Petersen cube is superior to the hypercube of comparable size with respect to usual parameters such as diameter and packing density. In addition, rings of all lengths and certain meshes, complete binary trees, X-trees, mesh of trees, tree machines and pyramids can be embedded into the folded Petersen cube with dilation and edge congestion of at most two [11,15,17].

[1] A 2-path is a path of length 2.