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Extending partial representations of subclasses of chordal graphs ☆

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ABSTRACT

Chordal graphs are intersection graphs of subtrees of a tree T. We investigate the complexity of the partial representation extension problem for chordal graphs. A partial representation specifies a tree T' and some pre-drawn subtrees of T'. It asks whether it is possible to construct a representation inside a modified tree T which extends the partial representation (i.e., keeps the pre-drawn subtrees unchanged).

We consider four modifications of T' leading to vastly different problems: (i) $T = T'$, (ii) T is a subdivision of T', (iii) T is a supergraph of T', and (iv) T' is a topological minor of T. In some cases, it is interesting to consider the complexity even when just T' is given and no subtree is pre-drawn. Also, we consider three well-known subclasses of chordal graphs: Proper interval graphs, interval graphs and path graphs. We give an almost complete complexity characterization. We further study the parametrized complexity of the problems when parametrized by the number of pre-drawn subtrees, the number of components of the input graph G and the size of the tree T'.

We describe an interesting relation with integer partition problems. The problem 3-Partition is used for all NP-completeness reductions. When the space in T' is limited, partial representation extension of proper interval graphs is "equivalent" to the BinPacking problem.

1. Introduction

Geometric representations of graphs and graph drawing are important topics of graph theory. We study *intersection representations* of graphs, where the goal is to assign geometrical objects to the vertices of a graph and encode edges by intersections of these objects. An intersection-defined class restricts the geometrical objects and contains all graphs representable by these restricted objects; for example, interval graphs are intersection graphs of closed intervals of the real

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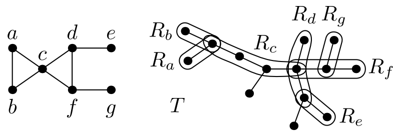

Fig. 1. An example of a chordal graph with one of its representations.

line. Intersection-defined classes have many interesting properties and appear naturally in numerous applications; for details see for example [2–4].

For a fixed class, its recognition problem asks whether an input graph belongs to this class; in other words, whether it has an intersection representation of this class. The complexity of recognition is well-understood for many classes; for example interval graphs can be recognized in linear-time [5,6].

We study a recently introduced generalization of the recognition problem called *partial representation extension* [7]. Given a graph and a partial representation (a representation of an induced subgraph), it asks whether it is possible to extend this partial representation to a representation of the entire graph. This problem falls into the paradigm of extending partial solutions, an approach that has been studied frequently in other circumstances. Often it proves to be much harder than building a solution from scratch, for example for graph coloring [8,9]. Surprisingly, the very natural problem of extending partially represented graphs was only considered recently.

The paper [7] gives an $\mathcal{O}(n^2)$-algorithm for interval graphs and an $\mathcal{O}(nm)$-algorithm for proper interval graphs. Also, several other papers consider this problem. Interval representations can be extended in time $\mathcal{O}(n+m)$ [10,11]. Proper interval representations can be extended in time $\mathcal{O}(n+m)$ and unit interval representations in time $\mathcal{O}(n^2)$ [12]. Polynomial-time algorithms are also described for function and permutation graphs [13], and for circle graphs [14]. The paper [15] describes minimal forbidden obstructions which make partial interval representations non-extendible.

In this paper, we follow this recent trend and investigate the complexity of partial representation extension of chordal graphs. Our mostly negative NP-completeness results are interesting since chordal graphs are the first class for which the partial representation extension problem is proved to be strictly harder than the original recognition problem. Also, we investigate three well-known subclasses – proper interval graphs, interval graphs and path graphs, for which the complexity results are richer. We believe that a better understanding of these simpler cases will provide tools to attack partial representation extension problems for chordal graphs (for example, from the point of the parameterized complexity). Also, one might get an insight into the classes of intersection graphs of connected subgraphs of fixed graphs H, which generalize chordal graphs, are very complicated and for instance contain circular-arc graphs.

1.1. Chordal graphs and their subclasses

A graph is *chordal* if it does not contain an induced cycle of length four or more, i.e., each "long" cycle is triangulated. The class of chordal graphs, denoted by CHOR, is well-studied and has many important properties. Chordal graphs are closed under induced subgraphs. Chordal graphs are perfect, and thus many hard combinatorial problems are easy to solve on chordal graphs: maximum clique, maximum independent set, k-coloring, etc. Their applications are even outside graph theory in matrix computations since chordal graphs possess the so-called *perfect elimination schemes* describing perfect reorderings of certain sparse matrices for the Gaussian elimination [4]. Chordal graphs can be recognized in time $\mathcal{O}(n+m)$ [16].

Chordal graphs have the following intersection representations [17]. For every chordal graph G, there exists a tree T and a collection $\{R_v \mid v \in V(G)\}$ of subtrees of T such that $R_u \cap R_v \neq \emptyset$ if and only if $uv \in E(G)$. For an example of a chordal graph and one of its intersection representations, see Fig. 1.

When chordal graphs are viewed as *subtrees-in-tree* graphs, it is natural to consider two other possibilities: *subpaths-in-path*, which gives *interval graphs* (INT), and *subpaths-in-tree*, which gives *path graphs* (PATH). For example, the graph in Fig. 1 is a path graph but not an interval one. Subpaths-in-path representations of interval graphs can be viewed as discretizations of real line representations. Interval graphs can be recognized in $\mathcal{O}(n+m)$ [5,6] and path graphs in time $\mathcal{O}(nm)$ [18,19].

In addition, we consider proper interval graphs (PINT). An interval graph is a *proper interval graph* if it has a representation $\mathcal{R}$ for which $R_u \subseteq R_v$ implies $R_u = R_v$; so no interval is a proper subset of another one.[1] Proper interval graphs can be recognized in time $\mathcal{O}(n+m)$ [20,21]. From the point of view of the problems we study, PINT behaves very similar to INT but there are subtle differences which we consider interesting. Also, partial representation extension of PINT is surprisingly very closely related to partial representation extension of unit interval graphs, considered in [12]; see Section 1.4 for details.

[1] It is possible to define proper interval graphs differently: If $R_u \subseteq R_v$, then $R_v \setminus R_u$ is empty or a connected subpath of T. In other words, no interval can be placed in the middle of another interval. Our results can be easily modified for this alternative definition.