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The role of planarity in connectivity problems parameterized by treewidth ☆

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ABSTRACT

For some years it was believed that for "connectivity" problems such as Hamiltonian Cycle, algorithms running in time $2^{O(\mathbf{tw})} \cdot n^{O(1)}$ – called *single-exponential* – existed only on planar and other topologically constrained graph classes, where **tw** stands for the treewidth of the n-vertex input graph. This was recently disproved by Cygan et al. [3, FOCS 2011], Bodlaender et al. [1, ICALP 2013], and Fomin et al. [11, SODA 2014], who provided single-exponential algorithms on general graphs for most connectivity problems that were known to be solvable in single-exponential time on topologically constrained graphs. In this article we further investigate the role of planarity in connectivity problems parameterized by treewidth, and convey that several problems can indeed be distinguished according to their behavior on planar graphs. Known results from the literature imply that there exist problems, like Cycle Packing, that *cannot* be solved in time $2^{o(\mathbf{tw} \log \mathbf{tw})} \cdot n^{O(1)}$ on general graphs but that can be solved in time $2^{O(\mathbf{tw})} \cdot n^{O(1)}$ when restricted to planar graphs. Our main contribution is to show that there exist natural problems that can be solved in time $2^{O(\mathbf{tw} \log \mathbf{tw})} \cdot n^{O(1)}$ on general graphs but that *cannot* be solved in time $2^{o(\mathbf{tw} \log \mathbf{tw})} \cdot n^{O(1)}$ even when restricted to planar graphs. Furthermore, we prove that Planar Cycle Packing and Planar Disjoint Paths *cannot* be solved in time $2^{o(\mathbf{tw})} \cdot n^{O(1)}$. The mentioned negative results hold unless the ETH fails. We feel that our results constitute a first step in a subject that can be further exploited.

1. Introduction

Motivation and previous work Treewidth is a fundamental graph parameter that, loosely speaking, measures the resemblance of a graph to a tree. It was introduced, among other equivalent definitions given by other authors, by Robertson and Seymour in the early stages of their monumental Graph Minors project [21], and its algorithmic importance was significantly popularized by Courcelle's theorem [2], stating that any graph problem that can be expressed in CMSO logic can be solved in time $f(\mathbf{tw}) \cdot n$ on graphs with n vertices and treewidth **tw**, where f is some function depending on the problem. Nevertheless, the function $f(\mathbf{tw})$ given by Courcelle's theorem is unavoidably huge [10], so from an algorithmic point of view it is crucial to identify problems for which $f(\mathbf{tw})$ grows *moderately* fast.

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Many problems can be solved in time $2^{O(\mathbf{tw}\log\mathbf{tw})}\cdot n^{O(1)}$ when the n-vertex input (general) graph comes equipped with a tree-decomposition of width **tw**. Intuitively, this is the case of problems that can be solved via dynamic programming on a tree-decomposition by enumerating all *partitions* or *packings* of the vertices in the bags of the tree-decomposition, which are $\mathbf{tw}^{O(\mathbf{tw})}=2^{O(\mathbf{tw}\log\mathbf{tw})}$ many. In this article we only consider this type of problems and, more precisely, we are interested in which of these problems can be solved in time $2^{O(\mathbf{tw})}\cdot n^{O(1)}$; such a running time is called *single-exponential*. This topic has been object of extensive study during the last decade. Let us briefly overview the main results on this line of research.

It is well known that problems that have *locally checkable certificates*,[1] like Vertex Cover or Dominating Set, can be solved in single-exponential time on general graphs. Intuitively, for this problems it is enough to enumerate *subsets* of the bags of a tree-decomposition (rather than partitions or packings), which are $2^{O(\mathbf{tw})}$ many. A natural class of problems that do *not* have locally checkable certificates is the class of so-called *connectivity problems*, which contains for example Hamiltonian Cycle, Steiner Tree, or Connected Vertex Cover. These problems have the property that the solutions should satisfy a *connectivity* requirement (see [1,3,24] for more details), and using classical dynamic programming techniques it seems that for solving such a problem it is necessary to enumerate partitions or packings of the bags of a tree-decomposition.

A series of articles provided single-exponential algorithms for connectivity problems when the input graphs are restricted to be topologically constrained, namely planar [9], of bounded genus [7,24], or excluding a fixed graph as a minor [8,23]. The common key idea of these works is to use special types of branch-decompositions (which are objects similar to tree-decompositions) with nice combinatorial properties, which strongly rely on the fact that the input graphs are topologically constrained.

Until very recently, it was a reasonable belief that (most) problems solvable in single-exponential time on general graphs should have locally checkable certificates, specially after Lokshtanov et al. [19] proved that one connectivity problem, namely Disjoint Paths, cannot be solved in time $2^{o(\mathbf{tw}\log\mathbf{tw})}\cdot n^{O(1)}$ on general graphs unless the Exponential Time Hypothesis (ETH) fails.[2] This credence was disproved by Cygan *et al.* [3], who provided single-exponential *randomized* algorithms on general graphs for several connectivity problems, like Longest Path, Feedback Vertex Set, or Connected Vertex Cover. More recently, Bodlaender et al. [1] presented single-exponential *deterministic* algorithms for basically the same connectivity problems, and an alternative proof based on matroids was given by Fomin et al. [11]. These results have been considered a breakthrough, and in particular they imply that most connectivity problems that were known to be solvable in single-exponential time on topologically constrained graph classes [7–9,23,24] are also solvable in single-exponential time on general graphs [1,3].

Our main results In view of the above discussion, a natural conclusion is that sparsity may not be particularly helpful or relevant for obtaining single-exponential algorithms. However, in this article we convey that sparsity (in particular, planarity) *does* play a role in connectivity problems parameterized by treewidth. To this end, among the problems that can be solved in time $2^{O(\mathbf{tw}\log\mathbf{tw})}\cdot n^{O(1)}$ on general graphs, we distinguish the following three disjoint types:

- **Type 1**: Problems that can be solved in time $2^{O(\mathbf{tw})}\cdot n^{O(1)}$ on general graphs.
- **Type 2**: Problems that *cannot* be solved in time $2^{o(\mathbf{tw}\log\mathbf{tw})}\cdot n^{O(1)}$ on general graphs unless the ETH fails, but that can be solved in time $2^{O(\mathbf{tw})}\cdot n^{O(1)}$ when restricted to planar graphs.
- **Type 3**: Problems that *cannot* be solved in time $2^{o(\mathbf{tw}\log\mathbf{tw})}\cdot n^{O(1)}$ even when restricted to planar graphs, unless the ETH fails.

Problems that have locally checkable certificates are of Type 1. As discussed in Section 3, known results [3,15] imply that there exist problems of Type 2, such as Cycle Packing. Our main contribution is to show that there exist (natural) problems of Type 3, thus demonstrating that some connectivity problems can indeed be distinguished according to their behavior on planar graphs. More precisely, we prove the following results:

- In Section 3 we provide some examples of problems of Type 2. Furthermore, we prove that Planar Cycle Packing cannot be solved in time $2^{o(\mathbf{tw})}\cdot n^{O(1)}$ unless the ETH fails, and therefore the running time $2^{O(\mathbf{tw})}\cdot n^{O(1)}$ is tight.
- In Section 4 we provide an example of problem of Type 3: Monochromatic Disjoint Paths, which is a variant of the Disjoint Paths problem on a vertex-colored graph with additional restrictions on the allowed colors for each path. To the best of our knowledge, problems of this type had not been identified before.

In order to obtain our results, for the upper bounds we strongly follow the algorithmic techniques based on *Catalan structures* used in [7–9,23,24], and for some of the lower bounds we use the framework introduced in [19], which has been also used in [3].

Additional results and further research We feel that our results about the role of planarity in connectivity problems parameterized by treewidth are just a first step in a subject that can be much exploited, and we think that the following avenues

[1] That is, certificates consisting of a constant number of bits per vertex that can be checked by a cardinality check and by iteratively looking at the neighborhoods of the input graph. See also [20] for a logical characterization of such problems.

[2] The ETH states that there exists a positive real number s such that 3-CNF-Sat with n variables and m clauses cannot be solved in time $2^{sn}\cdot(n+m)^{O(1)}$. See [18] for more details.