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Limit cycle structure for dynamic bi-threshold systems

Sichao Wu [a], Abhijin Adiga [a], Henning S. Mortveit [a,b,*]

[a] Network Dynamics and Simulation Science Laboratory, VBI, Virginia Tech, United States
[b] Department of Mathematics, Virginia Tech, United States

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ABSTRACT

In this paper we determine the limit cycle structures of synchronous and sequential finite dynamical systems governed by *dynamic* bi-threshold functions over non-uniform networks. So far, work in this area has been concerned with static threshold values, and our results generalize these. In particular, this includes the celebrated result by Goles and Olivos on synchronous neural networks (1981) [5], the work by Kuhlman et al. (2012) [12] on static bi-threshold systems, and the work by Chang et al. (2014) [2] on dynamic standard threshold systems.

In our work, the state of each vertex v includes the usual binary state $x_v \in \{0, 1\}$, the dynamic up-thresholds $k_v^{\uparrow}$ and down-thresholds $k_v^{\downarrow}$ whose values change only upon a transition of x_v. We show that the projection of the periodic orbits onto the Boolean x-dynamics have maximal size 2 under the synchronous update method regardless of how the up- and down-thresholds evolve. The proof is a careful extension of the technique originally developed in the proof by Goles and Olivos (1981) [5] and which was later modified by Kuhlman et al. (2012) [12] to cover bi-threshold systems. We also derive sufficient conditions on the evolution of the up- and down-thresholds that ensures that the sequential systems only have fixed points as limit sets. The results should be relevant for modeling and analyzing a large range of social and biological systems where agent adaptation occurs and needs to be accounted for.

1. Introduction

A large range of complex phenomena such as biological, social and technical systems, can be described as discrete, dynamical processes taking place over networks, see for example [10,11,3,7]. Many dynamical system models have been proposed to capture such systems. Examples include cellular automata (CA) which were originally introduced by von Neumann [18], Boolean networks (BN) proposed by Kauffman in [10,11], Random Boolean Networks (RBN), see [15,14], automata networks [4], and polynomial dynamical systems (PDS), see [8,9]. Our focus here is on dynamics evolving as

$$x(t+1) = F\big(x(t)\big) \quad \text{where } F\colon M \longrightarrow M \tag{1}$$

is a map on a suitable space M. In the case of Boolean networks and this paper we take $M = \{0, 1\}^n$ unless stated otherwise. States are denoted by $x = (x_1, \dots, x_n) \in \{0, 1\}^n$. The focus in this paper is on systems of the form (1) that are constructed from three components. The first is a finite graph X where each vertex v has a state x_v from a set K. For each vertex v

* Corresponding author at: Network Dynamics and Simulation Science Laboratory, VBI, Virginia Tech, United States.
E-mail addresses: sichao@vbi.vt.edu (S. Wu), abhijin@vbi.vt.edu (A. Adiga), henning.mortveit@vt.edu (H.S. Mortveit).

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there is a vertex function f_v that is used to map $x_v(t)$ to $x_v(t+1)$ based on the vertex states in the 1-neighborhood of v (including x_v) and thus governs the *local dynamics*. Third, an update method governs how the vertex functions assemble to a map F of the form (1) which produces the *global dynamics*. The results in this paper cover the parallel and the sequential update methods. From the point of view of modeling, the vertex state captures the status of a particular agent such as the health state in an epidemic model. The vertex function encodes the evolution of the agent's state as a function of its own state and those of its neighbors in the graph X. The update scheme governs how the collection of agent states is updated collectively in a time step.

This paper considers vertex functions that are a generalization of *bi-threshold functions*. The Boolean bi-threshold function $\theta_{v,k^{\uparrow},k^{\downarrow}}\colon \{0,1\}^N \longrightarrow \{0,1\}$ generalizes the standard threshold functions and is defined by

$$\theta_{v,k^{\uparrow},k^{\downarrow}}(x_1,\dots,x_N) = \begin{cases} 1, & \text{if } x_v = 0 \text{ and } \sum_{j=1}^{N} x_j \geqslant k^{\uparrow} \\ 0, & \text{if } x_v = 1 \text{ and } \sum_{j=1}^{N} x_j < k^{\downarrow} \\ x_v, & \text{otherwise,} \end{cases} \tag{2}$$

where the integers $k^{\uparrow}$ and $k^{\downarrow}$ are called the *up- and down-thresholds*, respectively. Here v is a designated vertex and $N = d(v)+1$ where $d(v)$ is the *degree* of v in X. We will consider non-uniform systems: the up- and down-thresholds will depend on the vertex, and we write $(k_v^{\uparrow})_v$ and $(k_v^{\downarrow})_v$ for the vertex-indexed sequences of up- and down-thresholds. We get the standard, static threshold systems when the up- and down-thresholds are the same.

For many applications, the up- and down-thresholds are not static, but are instead governed by the dynamics itself. In the case of epidemics such as malaria [16], for example, the acquired partial immunity will generally increase upon exposure to the parasite. From the point of view of modeling, this can be captured through an increase in the threshold $k^{\uparrow}$ upon the transition from susceptible (S or 0) to infected (I or 1). The same holds for influenza [3]. Similarly, thresholds related to addictive behavior such as smoking and re-smoking may change with the number of episodes that has taken place (see [17] and references therein). In the case of acquired partial immunity for malaria there is also a drop upon periods with no exposure. In this paper, we investigate the dynamics of several extended threshold models that can be used to capture such phenomena.

Contribution. In this paper we extend the results of [2,12,5] to bi-threshold graph dynamical systems where the thresholds $k^{\uparrow}$ and $k^{\downarrow}$ can evolve. In the first main result (Theorem 4), we extend the result of Goles and Olivos [5] for synchronous neural networks, see Section 4. The extension involves a careful refinement of the partition $\mathcal{C}$ developed in [5] and elaborated further in [12]. Perhaps somewhat surprisingly, the maximal length of a periodic orbit for such systems is 2. The second main result (Theorem 10) is for the sequential update mechanism and gives conditions on the dynamics of the up- and down-threshold that ensure the only limit sets are fixed points. The proof, which employs a potential function, also allows us to construct a bound for the maximal transient length.

Paper organization. In the next section, we give necessary background definitions and terminology. This is followed by some preliminary observations in Section 3. Section 4 characterizes the limit set structure of synchronous, neural networks with dynamic bi-threshold functions while Section 5 considers the transient structure of sequential systems and conditions for when they only have fixed points as attractors. We conclude with a summary in Section 6.

2. Preliminaries

Let X be a simple graph with vertex set $v[X] = \{1,2,3,\dots,n\}$. To each vertex v we assign a state x_v from a finite set M_v, and refer to this as the *vertex state*. The *system state* is an n-tuple $x = (x_1,x_2,\dots,x_n) \in M := \prod_v M_v$. Let $n[v]$ denote the sorted sequence of vertices from the 1-neighborhood of v in X, and let $x[v] \in \prod_{v' \in n[v]} M_{v'}$ denote the corresponding restriction of the system state $x = (x_1,x_2,\dots,x_n)$ to $n[v]$. We denote the *degree* of v by $d(v)$. Each vertex is assigned a *vertex function*

$$f_v\colon \prod_{v' \in n[v]} M_{v'} \longrightarrow M_v,$$

and an X*-local function* $F_v\colon M \longrightarrow M$ which is defined by

$$F_v(x) = \bigl(x_1,x_2,\dots,f_v\bigl(x[v]\bigr),\dots,x_n\bigr).$$

We use S_X to denote the set of permutations over the vertex set of X and write $\pi = (\pi_1,\pi_2,\dots,\pi_n)$ for the elements of S_X. We can now define sequential and synchronous dynamical systems which are both special cases of graph dynamical system (GDS).

Definition 1. Let X, $(f_v)_v$ and π be as above. The associated *sequential dynamical system* (SDS) map $F_\pi\colon M \longrightarrow M$ is the composition

$$F_\pi = F_{\pi_n} \circ \cdots \circ F_{\pi_1}.$$