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Three research directions in non-uniform cellular automata

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A B S T R A C T

The paper deals with recent developments about non-uniform cellular automata. After reviewing known results about structural stability we complete them by showing that also sensitivity to initial conditions is not structurally stable. The second part of the paper reports the complexity results about the main dynamical properties. Some proofs are shortened and clarified. The third part is completely new and starts the exploration of the fixed points set of non-uniform cellular automata.

1. Introduction and motivations

Last fifty years witness the growing interest of researchers in cellular automata (CA) both from the theoretical and applicative point of view. CA are indeed a formal model for complex systems [41,13,12,26]. They essentially consist of an infinite number of finite automata arranged on a regular lattice ($\mathbb{Z}$ in this paper). Each automaton takes a state chosen from a finite set. The state is updated at each time step according to a local rule on the basis of the state of the automaton itself and the ones of a fixed set of neighboring automata. All automata of the lattice apply the same local rule and have the same neighborhood pattern. All updates are synchronous. These few lines single out the three main characteristics of the model: locality, synchronicity and uniformity. Relaxing these properties originates variants of the model that have a great interest in their own, especially in practical applications. This paper surveys recent results about the dynamical behavior of non-uniform cellular automata (ν-CA), *i.e.*, those variants of CA in which each automaton can have a different local rule (and hence a possibly different neighborhood). In Section 3, the reader can immediately get convinced that ν-CA constitute a real stand-alone model. Indeed, many of the classical results concerning CA dynamics are disproved. For example, injective ν-CA are no longer necessarily surjective and expansive ν-CA do not need to be surjective.

Structural stability is one of the main motivations for the study of this new model. This notion has been introduced by Aleksandr Andronov and Lev Pontryagin in relation to the qualitative behavior of a dynamical system [4]. A property of a system is structurally stable if small perturbations of the system do not affect it. Therefore, the structural stability of a dynamical system is often interpreted as its robustness to failures.

In the context of cellular automata, there are several ways to model structural perturbations. The first possibility is to consider transient failures. In [30], Peter Gacs proposed a model in which each cell has some probability p of being updated and $1 - p$ of keeping its current state (*i.e.*, the identity local rule is applied). More recently, the case of transient failures

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is more viewed as a modification of the updating scheme for cellular automata. This turns into a model of asynchronous cellular automata. An ever-growing literature exists on this subject. The interested reader can begin with [43,27,17,18,16,28], for example.

Another possibility is to modify the topology of the lattice, *i.e.*, the links between cells can be rewired. As the neighborhood can vary, to be consistent with the definition of the local rule, this kind of perturbation is mainly used for totalistic rules, namely functions of the values in the cells wherever they are positioned [32].

This paper is more concerned with the case of permanent failures. Given a cellular automaton, the local rule is replaced in some positions by arbitrary local rules. The structural stability of several properties is investigated according to those perturbations. Section 4 shows that (among other things) neither sensitivity to the initial conditions nor almost equicontinuity are structurally stable properties for CA. Indeed, it seems that most of the dynamical properties are not structurally stable except for equicontinuity.

These results also point out that structural stability is influenced not only from the specific properties of the local rules perturbing the system but even from their relative distribution and cooperation. Section 5 tries then to characterize all the distributions of local rules inducing a given dynamical property. Indeed, if the radius and the set of states are fixed, the set of distinct local rules is finite. Therefore, a distribution of local rules is just a bi-infinite word over some finite alphabet and, seemingly, the set of distributions inducing a given dynamical property is nothing but a language of bi-infinite word. This simple remark leads us to characterize bi-infinite languages of distributions inducing ν-CA with some interesting dynamical property. The idea is that the complexity of the language quantifies, in a certain sense, the complexity of the dynamical property itself.

For example, we illustrate that distributions inducing surjective ν-CA form a sofic subshift while injective ν-CA are characterized by ζ-rational languages. Going more in deep along this direction seems difficult since one needs more properties on the intrinsic structure of the ν-CA to prove further results. For these reasons, we focused on linear non-uniform cellular automata, *i.e.*, ν-CA with an additive global rule (see [38,11] for the main results about additive CA). The additivity constraint allowed to show that for linear ν-CA also equicontinuity and sensitivity to the initial conditions are characterized by ζ-rational languages. Indeed, in the more general case, we know that none of those properties is decidable and hence the language cannot be ζ-rational.

The last part of the paper reports a recent research direction focusing on fixed points. These latter play a fundamental role in many modeling situations since they represent the viable/feasible ones (in systems biology, for example). Section 6 first writes down some more or less folklore results about the cardinality of the set of fixed points in CA. The proofs are essentially based on the well-known De Bruijn graphs. More or less the same results about cardinality can be restated also in the context of ν-CA. Moreover, we proved the following interesting characterization. Consider a set S of configurations which can represent a pointed ζ-rational language. Then, in the case of rν-CA (a subclass of ν-CA), the set of distributions having S as set of fixed points is a pointed ζ-rational language.

All the three research directions need further investigations and provide more questions than answers. Some of these questions are addressed in the last section.

2. Background

In this section, we briefly recall standard definitions about CA and discrete dynamical systems (see for instance [35,22,2,15,14,1,23,10] for introductory matter and recent results).

For all integers i and j with $i \le j$, let $[i, j]$ denote the set $\{i, i+1, \dots, j\}$. With the obvious meaning, we shall use the notations $]-\infty, i]$ and $[i, \infty[$.

Configurations and cellular automata Let A be a finite set (an *alphabet*). A *configuration* is a function from $\mathbb{Z}$ to A. The *configuration set* $A^{\mathbb{Z}}$ is usually equipped with the metric d defined as follows

$$\forall x, y \in A^{\mathbb{Z}}, \quad d(x, y) = 2^{-n}, \quad \text{where } n = \min\{i \in \mathbb{N}, x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}.$$

The set $A^{\mathbb{Z}}$ is a compact, totally disconnected and perfect topological space (*i.e.*, $A^{\mathbb{Z}}$ is a Cantor space). For all integers i and j with $i \le j$, and for all configurations $x \in A^{\mathbb{Z}}$, we denote by $x_{[i,j]}$ the word $x_i \cdots x_j \in A^{j-i+1}$, *i.e.*, the portion of x inside the interval $[i, j]$. Similarly, $u_{[i,j]} = u_i \cdots u_j$ is the factor of a word $u \in A^l$ inside $[i, j]$ (here, $i, j \in [0, l-1]$). For any word $u \in A^*$, $|u|$ denotes its length. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $[u]_i = \{x \in A^{\mathbb{Z}} : x_{[i,i+k)} = u\}$. Cylinders are clopen sets w.r.t. the metric d and they form a basis for the topology induced by d. A configuration x is said to be *a-finite* for some $a \in A$ if there exists $k \in \mathbb{N}$ such that $x_i = a$ for all $i \in \mathbb{Z}$, $|i| > k$. In the sequel, the collection of the a-finite configurations for a certain a will be simply called set of finite configurations.

A (one-dimensional) *cellular automaton* (CA) is a structure $\langle A, r, f \rangle$, where A is the *alphabet*, $r \in \mathbb{N}$ is the *radius* and $f : A^{2r+1} \to A$ is the *local rule* of the automaton. The local rule f induces a *global rule* $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as follows,

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(x)_i = f(x_{i-r}, \dots, x_{i+r}).$$

Recall that F is a uniformly continuous map w.r.t. the metric d.