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Solutions for the stable roommates problem with payments[☆]

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ABSTRACT

The stable roommates problem with payments has as input a graph $G = (V, E)$ with an edge weighting $w : E \to \mathbb{R}_{\geq 0}$ and the problem is to find a stable solution. A solution is a matching M with a vector $p \in \mathbb{R}^V_{\geq 0}$ that satisfies $p_u + p_v = w(uv)$ for all $uv \in M$ and $p_u = 0$ for all u unmatched in M. A solution is stable if it prevents blocking pairs, i.e., pairs of adjacent vertices u and v with $p_u + p_v < w(uv)$, or equivalently, if the total blocking value $\sum_{uv \in E} \max\{0, w(uv) - (p_u + p_v)\} = 0$. By pinpointing a relationship to the accessibility of the coalition structure core of matching games, we give a constructive proof for showing that every yes-instance of the stable roommates problem with payments allows a path of linear length that starts in an arbitrary unstable solution and that ends in a stable solution. This generalizes a result of Chen, Fujishige and Yang (2011) [4] for bipartite instances to general instances. We also show that the problems Blocking Pairs and Blocking Value, which are to find a solution with a minimum number of blocking pairs or a minimum total blocking value, respectively, are NP-hard. Finally, we prove that the variant of the first problem, in which the number of blocking pairs must be minimized with respect to some fixed matching, is NP-hard, whereas this variant of the second problem is polynomial-time solvable.

1. Introduction

Consider a group of tennis players participating in a doubles tennis tournament. Each two players estimate the expected prize money they could win together by forming a pair in the tournament. Moreover, each player can negotiate his share of the prize money with his chosen partner in order to maximize his own prize money. Can the players be matched together such that no two players have an incentive to leave the matching in order to form a pair together? This example has been given by Eriksson and Karlander [6] to introduce the stable roommates problem with payments.

The stable roommates problem with payments generalizes the stable marriage problem with payments [14] and can be modeled by a weighted graph $G = (V, E)$, i.e., that has an edge weighting $w : E \to \mathbb{R}_{\geq 0}$. A vector $p \in \mathbb{R}^V$ with $p_u \geq 0$ for all $u \in V$ is said to be a *matching payoff* if there exists a matching M in G, such that $p_u + p_v = w(uv)$ for all $uv \in M$, and $p_u = 0$

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for each u that is not incident to an edge in M. We then say that p is a payoff *with respect to* M, and we call the pair (M, p) a *matching with payoffs*. A pair of adjacent vertices $\{u, v\}$, i.e., with an edge between them, is a *blocking pair* of $p \in \mathbb{R}^V_{\geq 0}$ if $p_u + p_v < w(uv)$, and their *blocking value* with respect to p is defined as $w(uv) - (p_u + p_v)$. The latter value expresses to which extent $\{u, v\}$ is a blocking pair. We define the *set of blocking pairs* of a vector $p \in \mathbb{R}^V_{\geq 0}$ as

$$B(p) = \{\{u, v\} \mid uv \in E \text{ and } p_u + p_v < w(uv)\},$$

and we define the *total blocking value* of p as

$$b(p) = \sum_{uv \in E} \max\{0, w(uv) - (p_u + p_v)\}.$$

The problem Stable Roommates with Payments is that of testing whether a weighted graph allows a *stable solution*, i.e., a matching with payoffs (M, p) such that $B(p) = \emptyset$, or equivalently, $b(p) = 0$. This problem is well known to be polynomial-time solvable (cf. [6]); recently, an $O(nm + n^2 \log n)$ time algorithm for weighted graphs on n vertices and m edges has been given [3].

We consider two natural questions in our paper:

1. Can we gradually transform an unstable solution into a stable solution assuming that a stable solution exists?
2. Can we efficiently find solutions for no-instances that are "as stable as possible"?

Question 1 is of structural importance, as it will give us some insight into the coalition formation process. A sequence of solutions starting from an unstable one and ending in a stable one is called a *path to stability*; we give a precise definition later. Question 2 is of algorithmic nature and is relevant when we consider no-instances of Stable Roommates with Payments. In order to answer it, we generalize this problem in two different ways leading to the following two decision problems. Given a weighted graph G and an integer $k \geq 0$, the Blocking Pairs problem is to test whether G allows a matching payoff p with $|B(p)| \leq k$, and the Blocking Value problem is to test whether G allows a matching payoff p with $b(p) \leq k$.

Questions 1 and 2 have been studied in two closely related settings that are well known and formed a motivation for our study. The first related setting is similar to ours except that payments are not allowed. Instead, each vertex u in an (unweighted) graph $G(V, E)$ has a linear order on its neighbors expressing a certain preference. Then two adjacent vertices u and v form a *blocking pair* relative to a matching M if either u is not matched in M or else u prefers v to its partner in M, and simultaneously, the same holds for v. This leads to the widely studied problem Stable Roommates introduced by Gale and Shapley [7]. In this setting, the results are as follows. Answering a question by Knuth [12], Roth and Vande Vate [13] showed the existence of a path to stability for any yes-instance provided that the instance is bipartite. Later, their result was generalized by Diamantoudi et al. [5] to be valid for general instances. Abraham, Biró and Manlove [1] showed that the problem of finding a matching with a minimum number of blocking pairs is NP-complete; note that the problem Blocking Value cannot be translated to this setting, due to the absence of cardinal utilities.

The second related setting originates from cooperative game theory. A *cooperative game with transferable utilities* (TU-game) is a pair (N, v), where N is a set of n *players* and a *value function* $v : 2^N \to \mathbb{R}_{\geq 0}$ with $v(\emptyset) = 0$ defined for every *coalition* S, which is a subset of N. In a *matching game* (N, v), the set N of players is the vertex set of weighted graph G, and the value of a coalition S is $v(S) = \sum_{e \in M} w(e)$, where M is a maximum weight matching in the subgraph of G induced by S. The strong relationship between the two settings stems from the fact that finding a *core allocation*, i.e., a vector $x \in \mathbb{R}^N$ with $\sum_{u \in N} x_u = v(N)$ and $\sum_{u \in S} x_u \geq v(S)$ for all $S \subseteq N$ is equivalent to solving the Stable Roommates with Payments (cf. [6]). The algorithms of Béal et al. [2] and Yang [15] applied to an n-player matching game with a nonempty core (i.e. that have at least one core allocation) find a path to stability with lengths at most $(n^2 + 4n)/4$ and $2n - 1$, respectively. For matching games, the problems Blocking Pairs and Blocking Value are formulated as the problems that are to test whether a matching game (N, E) allows an allocation x with $|B(x)| \leq k$, or $b(x) \leq k$, respectively, for some given integer k. Biró, Kern and Paulusma [3] showed that the first problem is NP-complete and that the second is polynomial-time solvable by formulating it as a linear program.

Our Results. In Section 2, we prove a structural result that provides an affirmative answer to Question 1. We show that any unstable solution for a weighted n-vertex graph G that is a yes-instance of Stable Roommates with Payments allows a path to stability of length at most $2n$. This generalizes a structural result of Chen, Fujishige and Yang [4], who show the existence of a path to stability for the aforementioned stable marriage problem with payments, which corresponds to the case when G is bipartite. In Section 3 we prove a number of computational complexity results. We first answer Question 2 by proving that Blocking Pairs and Blocking Value are NP-complete. The latter result is somewhat surprising, as the corresponding problem is polynomial-time solvable for matching games; we refer to Table 1 for a survey. In addition, we show that Blocking Value does become polynomial-time solvable if the desired matching payoff is to be with respect to some specified matching M that is part of the input, whereas this variant of Blocking Pairs turns out to be NP-complete.

2. Paths to stability

We first give a useful lemma, which immediately follows from the aforementioned fact that finding a core allocation in a matching game (N, v) defined on a weighted graph $G = (N, E)$ is equivalent to finding a stable solution for G.