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Efficient sub-5 approximations for minimum dominating sets in unit disk graphs [☆]

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ABSTRACT

A unit disk graph is the intersection graph of n congruent disks in the plane. Dominating sets in unit disk graphs are widely studied due to their applicability in wireless ad-hoc networks. Because the minimum dominating set problem for unit disk graphs is **NP**-hard, numerous approximation algorithms have been proposed in the literature, including some PTASs. However, since the proposal of a linear-time 5-approximation algorithm in 1995, the lack of efficient algorithms attaining better approximation factors has aroused attention. We introduce an $O(n+m)$ algorithm that takes the usual adjacency representation of the graph as input and outputs a 44/9-approximation. This approximation factor is also attained by a second algorithm, which takes the geometric representation of the graph as input and runs in $O(n \log n)$ time regardless of the number of edges. Additionally, we propose a 43/9-approximation which can be obtained in $O(n^2 m)$ time given only the graph's adjacency representation. It is noteworthy that the dominating sets obtained by our algorithms are also independent sets.

1. Introduction

A *unit disk graph* G is a graph whose n vertices can be mapped to points in the plane and whose m edges are defined by pairs of points within Euclidean distance at most 1 from one another. Alternatively, one can regard the vertices of G as mapped to coplanar disks of unit diameter, so that two vertices are adjacent whenever the corresponding disks intersect.

A *dominating set* D is a subset of the vertices of a graph such that every vertex not in D is adjacent to some vertex in D. An *independent set* is a subset of mutually non-adjacent vertices. An *independent dominating set* is a dominating set which is also an independent set. Note that any maximal independent set is an independent dominating set.

Dominating sets in unit disk graphs are widely studied due to their application in wireless ad-hoc networks [15]. Since it is **NP**-hard to compute a minimum dominating set of a unit disk graph [4], several approximation algorithms have been proposed [6,7,11,12,15,19,23]. Such algorithms are of two main types. *Graph-based algorithms* receive as input the adjacency representation of the graph and assume no knowledge of the point coordinates, whereas *geometric algorithms* work in the Real RAM model of computation and receive solely the vertex coordinates as input.[1]

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[1] The Real RAM model is a technical necessity, otherwise storing the coordinates of the vertices would require an exponential number of bits [17].

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If the coordinates of the n disk centers are known, the m edges of the corresponding graph G can be figured out easily. It can be done in $O(n+m)$ time under the Real RAM model with floor function and constant-time hashing, and in $O(n \log n + m)$ time without those operations [1]. Thus, for the price of a conversion step, graph-based algorithms can be used when the input is a unit disk realization of G. However, unless $\mathbf{P} = \mathbf{NP}$, no efficient algorithm exists to decide whether a given graph admits a unit disk realization [3], let alone exhibit one. As a consequence, geometric algorithms cannot be efficiently transformed into graph-based algorithms. In this paper, we introduce approximation algorithms of both types, benefiting from the same approximation factor analysis. The proposed graph-based algorithm runs in $O(n+m)$ time, and the geometric algorithm runs in $O(n \log n)$ time regardless of m.

Previous algorithms. A graph-based 5-approximation algorithm that runs in $O(n+m)$ time was presented in [15]. The algorithm computes a maximal independent set, which turns out to be a 5-approximation because unit disk graphs contain no $K_{1,6}$ as induced subgraphs, as shown in that same paper.[2]

Polynomial-time approximation schemes (PTAS) were first presented as geometric algorithms [12] and later as graph-based algorithms [19]. Also, a graph-based PTAS for the more general disk graphs was proposed in [11]. Unfortunately, the complexities of the existing PTASs are high-degree polynomials. For example, the PTAS presented in [19] takes $O(n^{225})$ time to obtain a 5-approximation (using the analysis from [6]). Although its analysis is not tight, the running time is too high even for moderately large graphs. The reason is that these PTASs invoke a subroutine that verifies by brute force whether a graph admits a dominating set with k vertices. The verification takes $n^{O(k)}$ time, and it is unlikely that this can be improved (unless $\mathbf{FPT} = \mathbf{W[1]}$, as proved in [16]). Such a subroutine is applied to several subgraphs, and the value of k grows as the approximation error decreases. A similar strategy was used in [13] to obtain a PTAS for the minimum independent dominating set.

The lack of fast algorithms with approximation factor less than 5 was recently noticed in [6], where geometric algorithms with approximation factors of 4 and 3 and running times respectively $O(n^9)$ and $O(n^{18})$ were presented. While a significant step towards approximating large instances, those algorithms require the geometric representation of the graph, and their running times are still polynomials of rather high degrees. Linear and near-linear-time approximation algorithms constitute an active topic of research, even for problems that can be solved exactly in polynomial time, such as maximum flow and maximum matching [5,22].

It is useful to contrast the minimum dominating set problem with the maximum independent set problem. While a maximal independent set is a 5-approximation to both problems, it is easy to obtain a geometric 3-approximation to the maximum independent set problem in $O(n \log n)$ time [18]. In the graph-based version, a related strategy takes roughly $O(n^5)$ time, though. No similar results are known for the minimum dominating set problem.

The existing PTASs for the minimum dominating set problem in unit disk graphs are based on some packing constraints that apply to unit disk graphs.[3] One of these constraints is the *bounded growth property*: the size of an independent set formed by vertices within distance r of a given vertex, in a unit disk graph, is at most $(1+2r)^2$. Note, however, that the bounded growth property is not tight. For example, for $r = 1$, it gives an upper bound of 9 vertices where the actual maximum size is 5. Since the bounded growth property is strongly connected to the problem of packing circles in a circle, obtaining exact values for all r seems unlikely [9].

Our contribution. Our main result consists of the two approximation algorithms given in Section 3: a graph-based algorithm, which runs in linear $O(n+m)$ time, and its geometric counterpart, which runs in $O(n \log n)$ time in the Real RAM model, regardless of the number of edges. The approximation factor of both algorithms is 44/9. The strategy in both cases is to construct a 5-approximate solution using the algorithm from [15], and then perform local improvements to that initial dominating set. Our main lemma (Lemma 9) uses forbidden subgraphs to show that a solution that admits no local improvement is a 44/9-approximation. Since the dominating sets produced by our algorithms are independent sets, the same approximation factor holds for the independent dominating set problem.

Proving that a certain graph is *not* a unit disk graph (and is therefore a forbidden induced subgraph) is no easy feat.[4] We make use of an assortment of results from discrete geometry in order to prove properties of unit disk graphs that are interesting *per se*. For example, we use universal covers and disk packings to show that the neighborhood of a clique in a unit disk graph contains at most 12 independent vertices. These properties, along with a tighter version of the bounded growth property, are collected in Section 2, and allow us to show that certain graphs are not unit disk graphs. Consequently, the analyses of our algorithms employ a broader set of forbidden subgraphs which include, but are not limited to, the $K_{1,6}$.

Additionally, in Section 4, we show that a possible, somewhat natural refinement to our graph-based algorithm leads to a tighter 43/9-approximation, albeit for the price of an extra $O(n^2)$ multiplying factor in the time complexity of the algorithm.

[2] The graph $K_{1,q}$ consists of a vertex with q pendant neighbors.

[3] In *packing problems*, one usually wants to enclose non-overlapping objects into a recipient covering the greatest possible fraction of the recipient area.

[4] The fastest known algorithm to decide whether a given graph is a unit disk graph is doubly exponential [21].