Theoretical Computer Science 540–541 (2014) 103–114

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Second-Price Ad Auctions with Binary Bids and markets with good competition

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ARTICLE INFO

Keywords:
Approximation algorithms
Combinatorial auctions
Matchings

ABSTRACT

Given a bipartite graph $G = (U, V, E)$ with $U = \{1, \dots, n\}$, and a positive budget B_v for each v in V, a B-matching M in G is a second-price B-matching if, for every edge uv in M, there is an edge uw in E so that less than B_w edges $u'w$ with $u' < u$ belong to M. The *Second-Price Ad Auction with Binary Bids* (B2PAA) consists of, given G and B as above, finding a second-price B-matching in G as large as possible. The particular case of this problem where $B_v = 1$ for all v, called *Second-Price Matching* (2PM), is known to be APX-hard and there is a 2-approximation for it. We present a way to use this approximation and similar ones to approximate B2PAA. Also, we formalize the idea of a competitive market, present an improved approximation for 2PM on competitive markets, extend the inapproximability result for competitive markets and analyze the performance of an algorithm of Azar, Birnbaum, Karlin, and Nguyen for the online 2PM on competitive markets. Our improved approximation can also be used for B2PAA.

1. Introduction

An annual report by Econsultancy and SEMPO [1] estimated that the North American search engine marketing industry was worth US$14.6 billions in 2009, up from US$13.5 billions in 2008 and based on an 8% year-on-year growth. Indeed, sponsored search is one of the most profitable forms of advertising and generated roughly US$7 billions in revenue in 2005 [10]. The problem of allocating bidders' ads to ad slots in online search in a way that maximizes the search engine revenue is of great economical importance for companies like Google and Microsoft. Motivated by this, many recent papers present models and results for this type of problem.

One of the models considered in the literature is the so-called First-Price Ad Auctions, where there is only one slot for advertisement, each bidder has a budget and makes an offer for this slot for some of the keywords. At the arrival of each keyword, one has to choose one of the bidders interested in this keyword as the winner, and this bidder is charged his bid for having his ad displayed. One can find in the literature results for the online and the offline version of this problem. Recall that, for maximization problems, the approximation ratio of an algorithm (and the competitive ratio for an online algorithm) is the supremum over all instances of the ratio between the value of an optimal solution (found by an offline algorithm) and the value of the solution found by the algorithm. For the online version of the First-Price Ad Auctions, the best known competitive ratio is 2 [11] when there is no restriction on the values of the bids relative to the budgets. When the bids are much smaller than the budgets, Mehta et al. [12] and Buchbinder et al. [3] presented an algorithm that achieves an optimal competitive ratio of $e/(e-1)$. Devanur and Hayes [6] showed, for any $\epsilon > 0$, under some assumptions,

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1 Partially supported by CNPq 309657/2009-1, 475064/2010-0, and Project MaCLinC of NUMEC/USP.
2 Supported by grant 2009/00387-7, São Paulo Research Foundation (FAPESP) and Project MaCLinC of NUMEC/USP.

0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.08.022

a $(1+\epsilon)$-competitive algorithm for the case when the keywords arrive according to a random order. For the offline version of the First-Price Ad Auctions, the best known result is due to Chakrabarty and Goel [4] and Srinivasan [13]. They presented a 4/3-approximation and showed that it is NP-hard to approximate this version within 16/15.

Azar et al. [2] considered the Second-Price Ad Auctions problem. In this model, at the arrival of each keyword, one has to choose two interested bidders and the one among these two that offered more for the keyword is the winner and pays the offer of the other bidder. (This is like running a second-price auction for the ad slot with only these two bidders.) Their model tries to capture the fact that search engines usually use the Generalized Second Price mechanism [7,14] to decide the winners and the prices of the slots for a given keyword. In fact, Goel et al. [9] had previously considered an online variant of the Second-Price Ad Auctions. Their model differs from the one by Azar et al. because it allows a bidder to bid above his remaining budget even though he is charged at most his remaining budget.

Azar et al. [2] presented an (n/c)-approximation for the offline Second-Price Ad Auctions problem, where n is the number of keywords and c is the smallest ratio of the budget and the bid of a bidder. They also showed that it is NP-hard to approximate the offline problem within an $\omega(n)$ factor. Then they considered a simpler problem, called Second-Price Matching (2PM), where the bidders offers are zero or one and they have a budget of one, that is, their ads will be displayed for at most one keyword. They proved that the offline version of 2PM is APX-hard and presented a 2-approximation for this case. For the online version, they showed a lower bound of n for the competitive ratio of any deterministic algorithm (and an n-competitive deterministic algorithm), a lower bound of 2 for the competitive ratio of any randomized algorithm, and a 5.083-competitive randomized algorithm for the problem.

1.1. Our results

We consider the particular case of the Second-Price Ad Auctions where each bid is 0 or 1. We call it the **Second-Price Ad Auction with Binary Bids** (B2PAA). Note that B2PAA is the generalization of 2PM where bidders have an arbitrary budget. First we concentrate on the offline version of these problems. We show how to use algorithms for 2PM that find a solution by removing edges from a maximum matching in the bipartite graph of bidders and keywords, to obtain an approximation for B2PAA. Using the 2-approximation of Azar et al. [2] for 2PM, we derive a 2-approximation for B2PAA.

Next we focus on instances of ad auction problems with good competition. Let us call the number of bidders interested in a keyword u the *degree* of u. We consider a market competitive if the minimum degree of the keywords is at least some fixed integer δ. That is, there are at least δ bidders interested in each keyword. We first parameterize the previous analysis and results on 2PM in terms of δ.

We adapt the complexity proof from Azar et al. and show that 2PM remains APX-hard for fixed $\delta > 2$. It is interesting to note however that the inapproximability threshold obtained weakens (approaches 1) as δ grows. Also, we show that the 2-approximation of Azar et al. [2] for 2PM does not achieve a better ratio for competitive markets.

Then we present an approximation algorithm for 2PM that achieves a ratio better than 2 for $\delta \geqslant 4$. The ratio of our algorithm approaches 1.5 as δ grows. For $\delta \leqslant 3$, the ratio of our algorithm is 2, therefore it is in general at least as good as the algorithm of Azar et al., and better for competitive markets. Our algorithm, as the one by Azar et al., uses a maximum matching and can be used to solve B2PAA, and the ratio in competitive markets is also preserved.

While searching for a better algorithm for 2PM, we implemented ours as well as some variants, and experimented with them. Based on what we observed in our experiments, we suggest a variant of our algorithm that works empirically better. This variant uses a particular maximum matching. Naturally, we tried to prove a better ratio for this variant. However, we were able to show that, unfortunately, any algorithm that produces a second-price matching that is contained in a maximum matching will not achieve a ratio better than 2 for 2PM in general.

Finally we consider the online 2PM and show that the RankingSimulate algorithm [2] has competitive ratio of $(2^{\delta-1}/(2^{\delta-1}-1))(\sqrt{e}/(\sqrt{e}-1))$, which is better than 5.083 for $\delta \geqslant 3$.

In the next section, we introduce some notation and describe formally the problems we address. In Section 3, we consider the offline version of Second-Price Ad Auctions with Binary Bids and present an approximation for this problem that uses a specific type of algorithm for 2PM, such as our algorithm presented later. In Section 4 we discuss the inapproximability threshold for 2PM on competitive markets and in Section 5 we prove a lower bound on the ratio of the algorithm of Azar et al. [2] for competitive markets. In Section 6, we present an α-approximation algorithm with $\alpha = 2$ if $\delta = 2$ and $\alpha = \frac{3\delta-5}{2\delta-4}$ if $\delta > 2$ and in Section 7 we prove that this analysis is tight. The new approximation for 2PM can be used to solve B2PAA as described in Section 3, and its ratio is at most 2 and goes to 1.5 as δ grows. In Section 8, we describe a heuristic that empirically improves the solution found by our algorithm and we show that any algorithm that produces a second-price matching from a maximum matching has approximation ratio at least 2 for 2PM. Finally, in Section 9 we show the analysis of RankingSimulate on competitive markets and we present some conclusions in Section 10.

2. Model and notation

Let $[n]$ denote the set $\{1, \ldots, n\}$ and consider a bipartite graph $H = (U, V, E)$ where $U = [n]$ for some positive integer n. We call U the set of *keywords* and V the set of *bidders*. The order of the keywords is given by U, that is, keyword u appears before keyword u' if $u < u'$. An edge in H is an ordered pair (u, v) where $u \in U$ and $v \in V$, and it represents the fact that bidder v is interested in having his ad displayed at a search for keyword u. We simply say that v is interested in u. For a