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Axiomatizing weighted synchronization trees and weighted bisimilarity

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ABSTRACT

We consider regular synchronization trees weighted over a semiring and provide sound and complete axiomatizations of these trees and their weighted bisimulation equivalence classes. We prove that they can be both axiomatized by a finite number of identities relatively to the general axioms of the fixed point operation captured by the notion of iteration theories. We present infinite equational and finite quasi-equational axiomatizations.

1. Introduction

The synchronization trees studied in this paper arise by unfolding labeled multi-entry and multi-exit transition systems. The transition systems and hence the trees obtained will be quite general since they may contain hyper-edges $v \to (v_1, \ldots, v_n)$, where $v, v_1, \ldots, v_n$ are vertices, appropriately labeled by letters of a ranked alphabet Σ and carrying a weight in a monoid M. When transition systems are considered together with a notion of weighted bisimulation, this monoid M is the multiplicative reduct of a semiring S.

Some edges whose target is a single leaf may be labeled by an exit symbol $\mathsf{ex}_1, \mathsf{ex}_2, \ldots$ representing successful termination. In a tree $1 \to p$, only the first $p \geqslant 0$ exit symbols may occur. In addition to trees $1 \to p$, we will also consider trees $n \to p$ for all $n, p \geqslant 0$, which are n-tuples of trees $1 \to p$. In the case when all symbols in Σ are of rank 1 and M is the trivial monoid $\mathbf{1}$, our notion of synchronization tree basically agrees with the one defined in [44], see also [1,6].

By a natural notion of composition (defined via substitution), we may turn synchronization trees over Σ weighted in M into a category $\mathbf{ST}_\Sigma^M$ whose objects are the natural numbers $n \geqslant 0$. Since each object n is the n-fold coproduct of object 1 with itself, $\mathbf{ST}_\Sigma^M$ is a Lawvere theory [34,6]. Thus, $\mathbf{ST}_\Sigma^M$ comes with a pairing operation

$$(f : n \to p, g : m \to p) \mapsto \langle f, g \rangle : n + m \to p$$

which creates a natural isomorphism $\mathbf{ST}_\Sigma^M(n,p) \times \mathbf{ST}_\Sigma^M(m,p) \to \mathbf{ST}_\Sigma^M(n+m,p)$. Moreover, there is a natural sum operation turning each hom-set into a commutative monoid $(\mathbf{ST}_\Sigma^M(n,p), +, 0_{n,p})$, and there is natural action of M on synchronization trees. Thus, $\mathbf{ST}_\Sigma^M$ is an "M-grove theory" as defined in the paper.

The fixed point equation for a tree $f : n \to n + p$ is the equation

$$x = f \cdot \langle x, \mathbf{1}_p \rangle$$

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in the variable $x : n \to p$ (where $\mathbf{1}_p : p \to p$ is the identity). When f is "guarded", this fixed point equation has a unique solution $f^\dagger : n \to p$. Thus, $\mathbf{ST}^M_\Sigma$ is also equipped with a (partial) dagger or fixed point operation $\mathbf{ST}^M_\Sigma(n, n+p) \to \mathbf{ST}^M_\Sigma(n, p)$, $n, p \geqslant 0$.

It was argued in [6] that all fixed point operations in computer science satisfy at least the identities of "iteration theories", see also [41]. But since the theories $\mathbf{ST}^M_\Sigma$ possess an additional additive structure and an action of M, there are nontrivial identities combining the Lawvere theory operations and dagger with the additive structure and the M-action. Our aim is to give an account of these identities in terms of complete systems of axioms. By our main result, the equational theory of the categories $\mathbf{ST}^M_\Sigma$ possess a finite axiomatization relatively to the axioms of iteration theories. Since iteration theories themselves have many known axiomatizations, we obtain several complete axiomatizations of the equational theory of these categories. We will derive our relative completeness result from the characterization of the theory $\mathbf{RST}^M_\Sigma$ of "regular" synchronization trees in $\mathbf{ST}^M_\Sigma$ as the free theories in the class of all models of the axioms.

When M is the multiplicative reduct of a semiring S, then we may define a weighted bisimilarity relation on $\mathbf{ST}^M_\Sigma$ (extending the notion of probabilistic bisimilarity [33] and the weighted bisimilarity of [11]) denoted $\equiv_b$. In the case when S is the boolean semiring, our notion of weighted bisimilarity agrees with the classical Milner–Park notion of bisimilarity [35,39]. Since the relation $\equiv_b$ turns out to be a congruence with respect to all operations, we may form the quotient $\mathbf{BST}^S_\Sigma = \mathbf{ST}^M_\Sigma / \equiv_b$. In the second part of the paper we show that the equational theory of the categories $\mathbf{BST}^S_\Sigma$ can also be captured by a finite number of equational axioms relatively to iteration theories (at least when S is "atomistic"). We will derive this result from the characterization of the theories $\mathbf{BRST}^S_\Sigma = \mathbf{RST}^M_\Sigma / \equiv_b$ of weighted bisimilarity equivalence classes of regular trees as the free theories in the corresponding axiomatic class.

As mentioned above, synchronization trees arise as unfoldings of weighted transition systems. The weight structure is usually a semiring which gives rise to the semantic notion of bisimilarity. A simpler semantics, where the additive structure of the semiring plays no role, equates two transition systems if they unfold to the same tree. Our results also show that bisimulation semantics can be characterized over tree semantics by two simple axioms ensuring that the semiring action respects the additive structure of the semiring.

In the above treatment, the dagger operation is partial. In the last part of the paper we show that when S is an "iteration semiring" [6,25], then the dagger operation on $\mathbf{BST}^S_\Sigma$ and $\mathbf{BRST}^S_\Sigma$ may be turned into a totally defined operation and present the corresponding completeness results.

Our results provide generalizations of those in [6,9,12,35] which are concerned with the case when the monoid M is the trivial monoid $\mathbf{1}$ or the semiring S is the boolean semiring $\mathbb{B}$. For related treatments of probabilistic bisimilarity, see [42,2]. Probabilistic bisimilarity in conjunction with ordinary bisimilarity was axiomatized in [16,36]. The papers [35,42,16,36] all use variants of the unique fixed point rule, whereas [9,6,2] also contain fully equational axiomatizations.

All the results of this paper may be presented using other formalisms. For example, one could use (abstract) clones [15] equipped with an additive structure, an interpretation of M or S, and a fixed point operation. This essentially amounts to considering only the "scalar morphisms" $1 \to p$, $p \geqslant 0$, together with the a composition operation

$$(f, f_1, \ldots, f_p) \mapsto f \cdot \langle f_1, \ldots, f_p \rangle,$$

where $f : 1 \to p$ and $f_i : 1 \to q$, the sum operation on scalar morphisms, the interpretation of M or S, and the scalar dagger operation $f \mapsto f^\dagger$, where $f : 1 \to 1 + p$. Other possible formalisms would involve μ-expressions, or the "where expressions" or "let-rec" expressions borrowed from functional programming. However, it is a rather straightforward matter to translate the results of this paper into these formalisms, see the references.

The paper is organized as follows. In Section 2, we define algebraic theories and theories equipped with an additive structure and an action of a monoid M or a semiring S, called M-grove and S-grove theories, respectively. Then, in Section 3 we equip theories with a dagger operation and define (partial) Conway and iteration M-grove and S-grove theories. Section 4 is devoted to synchronization trees. In Theorem 4.1 we prove that synchronization trees over a ranked alphabet weighted in a monoid M form a partial iteration M-grove theory. Next, in Section 5, we provide various axiom systems for regular synchronization trees and the equational theory of synchronization trees. We derive our completeness results from Theorem 5.1 which gives a characterization of the theories of regular synchronization trees as the free partial iteration M-grove theories. In Section 6, we consider iteration S-grove theories of synchronization trees, where S is a generic semiring without any positivity properties. We define a notion of weighted bisimilarity which turns out to be a congruence with respect to the theory operations, sum, dagger, and the S-action. We thus obtain the partial iteration S-grove theories of weighted bisimilarity equivalence classes of synchronization trees and regular synchronization trees. In Theorem 7.7, we characterize the theories of weighted bisimilarity equivalence classes of regular trees as the free partial iteration S-grove theories for "atomistic" semirings. Moreover, in Theorem 7.4, we prove that for all semirings S, the theories of weighted bisimilarity equivalence classes of regular trees are the free partial iteration S-grove theories satisfying an extra condition (weak or strong functoriality). Our freeness results yield complete axiomatizations of weighted bisimilarity over regular trees and the equational theory of weighted bisimilarity. The dagger operation in the above results is a partial operation. In Section 8, we show that the dagger operation may be turned into a completely defined operation whenever S is an iteration semiring, and we derive several freeness and axiomatization results in this setting.