Theoretical Computer Science 532 (2014) 80–90

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Succinct strictly convex greedy drawing of 3-connected plane graphs

Jiun-Jie Wang *, Xin He

Department of Computer Science and Engineering, State University of New York at Buffalo, Buffalo, NY 14260, United States

ARTICLE INFO

Keywords:
3-connected plane graphs
Succinct
Convex greedy drawing

ABSTRACT

Geometric routing by using virtual locations is an elegant way for solving network routing problems. Greedy routing, where a message is simply forwarded to a neighbor that is closer to the destination, is a simple form of geometric routing. A greedy drawing of a graph G is a drawing of G for which the greedy routing works. Papadimitriou and Ratajczak conjectured that every 3-connected plane graph has a greedy drawing on the $\mathcal{R}^2$ plane (Papadimitriou and Ratajczak, 2005 [9]). Leighton and Moitra settled this conjecture positively in Leighton and Moitra (2010) [8]. A similar result was obtained by Angelini et al. (2010) [2]. However, their drawings have two major drawbacks: (1) their drawings are not necessarily planar; and (2) $\Omega(n \log n)$ bits are needed to represent each coordinate of their drawings, which is too large for routing algorithms for certain networks. Recently, He and Zhang [7] showed that every triangulated plane graph has a succinct (using $O(\log n)$ bit coordinates) greedy drawing on the $\mathcal{R}^2$ plane with respect to a metric function based on Schnyder realizers. However, their method does not work for 3-connected plane graphs. In this paper, we show that every 3-connected plane graph has a drawing on the $\mathcal{R}^2$ plane that is succinct, planar, strictly convex, and is greedy with respect to a metric function based on parameters derived from Schnyder woods.

1. Introduction

As communication technology progresses, traditional wired communication networks are rapidly replaced by wireless networks (such as sensor networks). The nodes of such networks are equipped with very limited memory and computing power. Thus the traditional network communication protocols are not suitable for them. *Geometric routing* is an interesting class of routing algorithms for wireless networks which use the geographic locations of the network nodes to determine routing paths. The simplest geometric routing is the *greedy routing*: to send a message from a source node s to a destination node t, s simply forwards the message to a neighbor that is closer to t.

However, greedy routing has drawbacks: the nodes need to be equipped GPS devises in order to determine their geographic locations, which are too expensive and power consuming. Even worse, a node s might be located in a *void position*, (namely s has no neighbor that is closer to the destination). In this case, the greedy routing completely fails. As a solution, Papadimitriou et al. [9] introduced the concept of *greedy drawing*: Instead of using real geographic coordinates, one could use graph drawing to compute the drawing coordinates for the nodes of a network G. Then the geometric routing algorithms rely on the drawing coordinates to determine the routing paths. Simply speaking, a *greedy drawing* is a drawing of G for which the greedy routing works. More precisely:

* Corresponding author.
E-mail addresses: jiunjiew@buffalo.edu (J.-J. Wang), xinhe@buffalo.edu (X. He).

0304-3975/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.tcs.2013.05.024

Definition 1. (See [9].) Let S be a set and $H(*,*)$ be a metric function over S. Let $G=(V,E)$ be a graph.

1. A *drawing* of G into S is a mapping $d: V \rightarrow S$ such that $u \neq w$ implies $d(u) \neq d(w)$.
2. The drawing d is *greedy* with respect to H if for any two vertices u, w of G $(u \neq w)$, u has a neighbor v such that $H(d(u), d(w)) > H(d(v), d(w))$.
3. The drawing d is *weakly greedy* with respect to H if for any two vertices u, w of G $(u \neq w)$, u has a neighbor v such that $H(d(u), d(w)) \geqslant H(d(v), d(w))$.

The following conjecture was posed in [9]:

Greedy Embedding Conjecture. Every 3-connected plane graph has a greedy drawing on the Euclidean plane $\mathcal{R}^2$.

Leighton and Moitra [8] recently settled this conjecture positively. A similar result was obtained by Angelini et al. [2]. However, their greedy drawings have drawbacks. First, as pointed out in [6], their drawings are not necessarily planar. Second, $\Omega(n\log n)$ bits are needed to represent each coordinate in the drawings produced by their algorithms. This is the same space usage required by the traditional routing table approaches, which is not practical for certain networks. A drawing is called *succinct* if each drawing coordinate can be represented by using $O(\log n)$ bits. To make the greedy routing scheme work in practice, we need a succinct greedy drawing.

Some progresses were made recently towards this goal. Goodrich et al. [6] used a set of virtual coordinates from the structure of *Christmas cactus* [8] to represent the vertices positions. However, their succinct representation of each vertex is totally different from the real underlying geometric embedding. Hence it is not a true greedy drawing. He and Zhang [7] showed that the classical Schnyder drawing of triangulated plane graphs is a succinct greedy drawing on the $\mathcal{R}^2$ plane with respect to a simple and natural metric function H. They also showed that the Schnyder drawing for 3-connected plane graphs is succinct, planar, and *weakly greedy* with respect to the same function H. With a greedy drawing, the greedy routing algorithm is very simple: the source node s just sends the message to a neighbor that is strictly closer to the destination t. With a weakly greedy drawing, the situation is more complicated. Since s may only have a neighbor u whose distance to t is the same, s may have to send the message to u. Hence the message might be sent back and forth among nodes with equal distances to t, but never reaches t.

Another desirable property of a drawing is *convex (strictly convex, respectively)*, in which every face of the graph is convex (strictly convex, respectively). Papadimitriou and Ratajczak also posed the following related conjecture [9]:

Convex Greedy Embedding Conjecture. Every 3-connected plane graph has a convex greedy drawing in the Euclidean plane.

Cao et al. [4] recently showed that there exists a 3-connected plane graph G such that any convex greedy drawing of G on the $\mathcal{R}^2$ Euclidean plane must use $\Omega(n)$-bit coordinates. Thus it is impossible to find a succinct, convex, greedy drawing on the $\mathcal{R}^2$ Euclidean plane for 3-connected plane graphs. In other words, in order to find a drawing on the $\mathcal{R}^2$ plane that is succinct, greedy, and convex, one must give up Euclidean distance.

In this paper, we describe a drawing for 3-connected plane graphs on the $\mathcal{R}^2$ plane that is succinct, strictly convex, and greedy with respect to a metric function based on *Schnyder woods*. In our drawing, each vertex is drawn at a grid point of a polynomial size grid, thus only $O(\log n)$ bits are needed to represent each drawing coordinate.

The classical Schnyder drawing of 3-connected plane graphs [5,10,11] is based on three *Schnyder coordinates*, which are obtained by counting the number of the faces in the three regions defined by Schnyder woods. It was shown in [7] that this drawing is only weakly greedy with respect to the metric function $H(u,v)$ defined in [7]. Our new drawing algorithm is also based on Schnyder woods. However, in addition to the three Schnyder coordinates, our algorithm also uses other information obtained from Schnyder woods, which we call *Schnyder parameters*. (There are totally 9 parameters, including three Schnyder coordinates.) These parameters are used to calculate the drawing coordinates and to define the metric function.

The paper is organized as follows. In Section 2, we give the definitions and the basic properties of Schnyder woods. In Section 3, we describe the metric function $H(u,v)$ used in our greedy drawing. We also show how to use Schnyder parameters to determine the relative locations of the vertices. In Section 4, we show that our drawing has the greedy property with respect to the metric function $H(u,v)$. In Section 5, we describe how to obtain the drawing coordinates from Schnyder parameters and show that the drawing is planar and strictly convex. Section 6 concludes the paper.

2. Preliminaries

Definitions not mentioned here are standard.

Definition 2. (See [1,5].) Let G be a 3-connected plane graph with three external vertices v_1, v_2, v_3 in counterclockwise (ccw) order. A *Schnyder wood* of G is a triple of rooted spanning trees $\{T_1, T_2, T_3\}$ of G with the following properties:

- For $i \in \{1,2,3\}$, the root of T_i is v_i. The edges of G are directed towards the root of T_i.