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The complexity of multicut and mixed multicut problems in (di)graphs

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ABSTRACT

The multicut problem with respect to the fixed pattern (di)graph Q is the following: given a (di)graph G, an integer k and a 1-1 mapping π of the vertices of Q to a subset X of $V(G)$; determine whether there exists a set S of at most k edges (arcs) of G such that $G-S$ has no $(\pi(q),\pi(q'))$-path for every edge (arc) qq' of Q. We first prove a dichotomy result for the complexity (they are all either polynomial or NP-complete) for multicut problems in (di)graphs in terms of the pattern Q. Then we consider a variant where the pattern is a mixed graph M; the input is a digraph D, an integer k and a 1-1 mapping π from $V(M)$ to some subset of $V(D)$ and the goal is to decide whether there exists a set S of at most k arcs of D that meets every $(\pi(q),\pi(q'))$-path in D for every arc qq' of M and every $(\pi(p),\pi(p'))$-path in the underlying undirected graph of D for every edge pp' of M. Again we prove a dichotomy result for the complexity of such mixed multicut problems in terms of the pattern M. It turns out that the only polynomial cases are those where the pattern is either a polynomial graph pattern or a polynomial digraph pattern. As soon as M contains both edges and arcs, the M-MIXED CUT problem becomes NP-complete.
Finally, we prove that the directed feedback arc set problem is NP-complete, even in the class of digraphs which have a feedback vertex set of size 2.

1. Introduction

Multicut is a common name for a class of problems of the following type: Given a (di)graph G with a weight function on the edges (arcs) and a set $\mathcal{S}=\{s_1,s_2,\dots,s_p,t_1,t_2,\dots,t_p\}$ of vertices in G which may contain repetitions but $s_i\neq t_i$ for $i\in[p]$; find a minimum weight subset E' of edges (arcs) of G such that $G-E'$ has no (s_i,t_i)-path for any $i\in[p]$. A well studied special case is the multiterminal problem where we are given distinct vertices $s_1,s_2,\dots,s_p$ of G and want to delete the minimum weight subset of edges (arcs) of G such that the resulting (di)graph has no (s_i,s_j)-path for any $i\neq j$. Both the multicut and the multiterminal problem have numerous practical applications in areas such as databases, routing and railroad transportation. For a survey on multicut and similar problems see [4]. These problems are generally NP-complete even for small values of p [5–7], see also [3].

In this paper we first classify (in terms of the prescribed pattern (di)graph) directed and undirected multicut problems in terms of being polynomially solvable or NP-complete. To be more precise: we consider the problems as defined by a prescribed pattern (di)graph Q (one problem for each Q). For a fixed pattern Q the input is an edge (arc) weighted (di)graph G, an integer k and a 1-1 mapping π from Q to a subset $X\subseteq V(G)$ and the goal is to decide whether there exists

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a set of edges (arcs) E' of G of total weight at most k such that $G - E'$ has no $(\pi(q_i), \pi(q_j))$-path for every edge (arc) q_iq_j of Q. We prove dichotomy theorems for both directed and undirected Q.

Then we consider a (seemingly) new class of problems for digraphs, where the pattern Q is now a mixed graph[1] and for a given bijection π from Q to a subset X of the vertex set of the input (arc weighted) digraph D and an integer k we must decide whether D has a set of arcs A' of total weight at most k, such that $D - A'$ has no $(\pi(q_i), \pi(q_j))$-path for every arc q_iq_j of Q and for every undirected edge q_aq_b in Q there is no $(\pi(q_a), \pi(q_b))$-path in the underlying graph of D (that is, even if we may violate orientations of arcs, we still cannot find a path from $\pi(q_a)$ to $\pi(q_b)$ that avoids A'). Clearly, as graphs and digraphs are also mixed graphs, all the NP-complete pattern graphs still give NP-complete problems, so the question is whether there are mixed patterns which are polynomial. As we shall see in this paper, there are none: as soon as the pattern contains both an edge and an arc the corresponding multicut problem is NP-complete.

2. Terminology and preliminaries

We refer to [2] for terminology on graphs. Terminology on digraphs not introduced here is consistent with [1]. We denote the vertex set and arc set of a digraph D by $V(D)$ and $A(D)$, respectively and write $D = (V, A)$ where $V = V(D)$ and $A = A(D)$. Paths and cycles in digraphs are always directed unless otherwise specified.

An (s,t)-path in a digraph D is a directed path from the vertex s to the vertex t. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and keeping multiple edges. A digraph D is **connected** if $UG(D)$ is a connected graph. When xy is an arc of D we say that x **dominates** y.

$\vec{K}_{a,b}$ denotes the orientation of the complete bipartite graph $K_{a,b}$, where all arcs are oriented from the partite set of size a to the partite set of size b. $D[X]$ denoted the digraph induced by the vertex set X in D.

We define the following three problems.

> Q-DIRECTED CUT.
> The digraph Q with vertex set $\{q_1, q_2, \ldots, q_l\}$ and arc set $A(Q)$ is considered fixed.
> **Input:** An integer k, a digraph $D = (V, A)$ together with a vertex set $X = \{x_1, x_2, \ldots, x_l\} \subseteq V$ $(|X| = |V(Q)|)$ and a non-negative integer-valued weight function ω on A.
> **Output:** Decide if there is a set of arcs, $S \subseteq A(D)$, such that $\omega(S) \leqslant k$ and for all $i, j \in \{1, 2, \ldots, l\}$ with $q_iq_j \in A(Q)$, there is no (x_i, x_j)-path in $D - S$.

> H-UNDIRECTED CUT.
> The graph H with vertex set $\{h_1, h_2, \ldots, h_l\}$ is considered fixed.
> **Input:** An integer k, a graph $G = (V, E)$ together with a vertex set $X = \{x_1, x_2, \ldots, x_l\} \subseteq V$ $(|X| = |V(H)|)$ and a non-negative integer-valued weight function ω on E.
> **Output:** Decide if there is a set of edges, $S \subseteq E(G)$, such that $\omega(S) \leqslant k$ and for all $1 \leqslant i < j \leqslant l$ with $h_ih_j \in E(H)$, there is no (x_i, x_j)-path in $G - S$.

> M-MIXED CUT.
> The mixed graph M with vertex set $\{m_1, m_2, \ldots, m_l\}$ is considered fixed.
> **Input:** An integer k, a digraph $D = (V, A)$ together with a vertex set $X = \{x_1, x_2, \ldots, x_l\} \subseteq V(D)$ $(|X| = |V(M)|)$ and a non-negative integer-valued weight function ω on A.
> **Output:** Decide if there is a set of arcs, $S \subseteq A(D)$, with $\omega(S) \leqslant k$, such that the following holds. If $m_im_j \in A(M)$ then there is no (x_i, x_j)-path in $D - S$, and if $m_im_j \in E(M)$, then there is no (x_i, x_j)-path in $UG(D - S)$.

Note that whenever we consider a problem M-MIXED CUT we may assume that M does not contain an arc uv such that uv is also an undirected edge of M. In the proofs below we will often add arcs or edges of weight $k + 1$, which has the effect that in any solution to any of the above problems such arcs or edges cannot belong to S.

Furthermore note that if all wights are positive integers and bounded by a polynomial in the size of the graph, then the problem can be transformed into an unweighted instance. This is done by, for each edge of weight d, adding d parallel edges of weight one. Furthermore if we do not want parallel edges, these d edges can be subdivided, such that we get d paths of length two.

The following results will be used in our proofs. The digraphs D_1, D_2 are depicted in Fig. 1.

Theorem 2.1. *(See [6].) D_1-DIRECTED CUT is NP-complete even when $\omega \equiv 1$, that is, all weights are equal to one.*

Theorem 2.2. *(See [7].) D_2-DIRECTED CUT is NP-complete even when the input is an acyclic digraph.*

[1] A mixed graph, M, consists of a vertex set $V(M)$ and an edge set (possibly empty), $E(M)$, of undirected edges and an arc set (possibly empty), $A(M)$, of directed arcs.