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Partitioning powers of traceable or hamiltonian graphs ☆

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ABSTRACT

A graph $G = (V, E)$ is arbitrarily partitionable (AP) if for any sequence $\tau = (n_1, \ldots, n_p)$ of positive integers adding up to the order of G, there is a sequence of mutually disjoint subsets of V whose sizes are given by τ and which induce connected graphs. If, additionally, for given k, it is possible to prescribe $l = \min\{k, p\}$ vertices belonging to the first l subsets of τ, G is said to be AP + k.

The paper contains the proofs that the kth power of every traceable graph of order at least k is AP + $(k-1)$ and that the kth power of every hamiltonian graph of order at least $2k$ is AP + $(2k-1)$, and these results are tight.

1. Introduction

Consider a simple graph $G = (V, E)$ of order n. A sequence $\tau = (n_1, \ldots, n_p)$ of positive integers is called *admissible* for G if it is a *partition* of n, *i.e.*, $n_1 + \cdots + n_p = n$. If additionally there exists a partition $(V_1, \ldots, V_p)$ of the vertex set V such that each V_i induces a connected subgraph of order n_i in G, then we say that τ is *realizable* in G, while $(V_1, \ldots, V_p)$ is called a *realization* of τ in G. If every admissible sequence is also realizable in G, then we say that this graph is *arbitrarily partitionable* (or *arbitrarily vertex decomposable*) and we call it an *AP graph* for short.

The notion of AP graphs was first introduced by Barth, Baudon and Puech in [2], and motivated by the following problem in computer science. Consider a network connecting different computing resources; such a network is modelled by a graph. Suppose there are p different users, where the ith one needs n_i resources from our network. The subgraph induced by the set of resources attributed to each user should be connected and each resource should be attributed to one user. So we are seeking a realization of the sequence $\tau = (n_1, \ldots, n_p)$ in this graph. Suppose that we want to do it for any number of users and any sequence of request. Thus, such a network should be an AP graph.

Independently (see [7] or [9]), this problem was also considered as a natural analogy of the similar notion in which vertices are replaced by edges (see for instance [1] or [8]).

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers. Obviously, a graph needs to be connected in order to be AP. The investigation of AP trees gained lots of attention in this context, since a connected graph is AP if one of its spanning trees is AP. It turned out, however, that the structure of AP trees is not obvious in general (see for instance [3–5] or [14]).

Since each traceable (*i.e.* containing a hamiltonian path) graph is evidently AP, each condition implying the existence of a hamiltonian path in a graph also implies that the graph is AP. So, AP graphs may be considered as a generalization of traceable (or hamiltonian) graphs (see for instance [10]).

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Suppose now that as managers of the computer network we have a number of at most k specially privileged clients (users), so-called *vip's*, each of whom may choose one computing resource which must be attributed to their connected subnetwork. It might be a powerful or conveniently located computer, which may serve our vip as an administrative center for managing the subnetwork. Then we naturally obtain the following modification of our model on graphs: Let $G = (V, E)$ be a graph of order n and let $n > k$. The graph G is said to be *AP + k* if for any partition $\tau = (n_1, \dots, n_p)$ of n and any sequence $(u_1, \dots, u_{k'})$ of k' pairwise distinct vertices of G with $k' \leqslant \min\{k, p\}$, there exists a realization $(V_1, \dots, V_p)$ of τ in G such that $u_1 \in V_1, \dots, u_{k'} \in V_{k'}$.

Observe that we have adopted the convention that the numbers representing the sizes of subnetworks attributed to vip's are listed in the beginning of the sequence τ.

If the number of subnetworks (users) is limited, say by r, *i.e.* we can realize in G each sequence $\tau = (n_1, \dots, n_p)$ with $p \leqslant r$, we say that G is r-AP. So, a graph is AP if it is r-AP for $r = 1, 2, \dots$ (see [12,13] and [15] for algorithmic approach for small k).

If additionally for a given $s \leqslant r$, each of the first s' users for any $s' \leqslant \min\{s, p\}$ is allowed to choose a vertex belonging to their subnetwork, then the corresponding graph G of order $n > r$ is called r-AP + s.

The most significant result concerning these notions is the following famous result on k-AP + k graphs by Győri [6] and, independently, Lovász [11].

Theorem 1. *Every k-connected graph G is k-AP + k.*

It is straightforward to notice that the converse is also true. Indeed, removal of $k-1$ vertices $v_1, \dots, v_{k-1}$ cannot disconnect a k-AP + k graph G, since otherwise there would not exist a realization $(V_1, \dots, V_k)$ of an admissible sequence $(1, \dots, 1, n-k+1)$ in G such that $v_1 \in V_1, \dots, v_{k-1} \in V_{k-1}$.

Analogously, by analyzing an admissible sequence $(1, \dots, 1, n-k)$, one can easily see that the following observation holds.

Observation 2. *Every AP + k graph has to be $(k+1)$-connected.*

It is worth noting that if we change the requirement concerning the number of parts we partition our network into (from bounded to arbitrary case), this may have dramatical consequences. For instance, consider the complete bipartite graph $K_{k,k}$. Since it is k-connected, then by Theorem 1, it is also k-AP + k. On the other hand, if we remove two vertices on the "same side" of $K_{k,k}$, we obtain the graph $K_{k,k-2}$, which evidently does not contain a perfect matching. In other words, with the above choice of two vip's, the sequence $(1, 1, 2, \dots, 2)$ is not realizable. In consequence, the graph $K_{k,k}$ is not even AP + 2.

Given a graph $G = (V, E)$, its kth power G^k is the graph obtained from G by adding the edge between every pair of vertices with distance at most k in G. In this paper we prove that kth powers of traceable graphs are AP + $(k-1)$, see Corollary 7, and that kth powers of hamiltonian graphs are AP + $(2k-1)$, see Corollary 9. These results are sharp.

2. Results

Given a path P_n (or a cycle C_n), its consecutive vertices $v_1, v_2, \dots, v_n$ define a natural *orientation* of the path (or the cycle). We shall call them also the *consecutive* vertices of its kth power P_n^k (or C_n^k). Similarly, v_1 and v_n will be called the *first* and the *last* vertices of P_n^k (C_n^k), respectively.

In both cases, for a vertex x, we shall also use the notation x^+ and x^- in order to denote the next or the previous vertex to x, respectively, with respect to the natural orientation. For two vertices a and b of the cycle C_n, we denote by aC_nb the set of all consecutive vertices of C_n starting from a and ending at b with respect to the natural orientation of the cycle.

First, we prove that kth powers of paths are AP + $(k-1)$. We shall use Lemma 5 below, which is even stronger than required for this purpose. The both results however will be then necessary to show that kth powers of cycles are AP + $(2k-1)$. Since the property of being AP + k is monotone with respect to adding edges, the results for paths and cycles immediately imply the corresponding properties for traceable and hamiltonian graphs, *i.e.*, Corollaries 7 and 9. Note here also that our results for paths (hence also for the family of traceable graphs) and for cycles (thus for hamiltonian graphs) are tight, since the connectivity of the kth power of a path P_n, $n \geqslant k+1$, is k, and the connectivity of the kth power of a cycle C_n, $n \geqslant 2k+1$, is $2k$. This is obvious for paths, while for cycles it is sufficient to notice that so that we could disconnect two vertices u, v of C_n^k, these must be at distance more than k in C_n. Then we have to remove (at least) k consecutive vertices from each of the two paths joining u and v in C_n.

Below we state two basic observations concerning the operation of removing a vertex from a graph $G = P_n^k$ being the kth power of a path P_n. Let $v_1, \dots, v_n$ be the consecutive vertices of P_n. By a graph obtained by removing the first (respectively, the last) vertex of G we mean the graph $G \setminus \{v_1\}$ (respectively, $G \setminus \{v_n\}$) with consecutive vertices given by $v_2, \dots, v_n$ or $v_1, \dots, v_{n-1}$, respectively. By a graph obtained by removing other than the first or the last vertex of G, say x, we mean the graph $G \setminus \{x\}$ with consecutive vertices given by $v_1, \dots, x^-, x^+, \dots, v_n$.

Observation 3. *A graph obtained by removing the first or the last vertex of any kth power of a path is also a kth power of a path.*