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Reversals and palindromes in continued fractions

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Abstract

Several results on continued fractions expansions are on indirect consequences of the *mirror formula*. We survey occurrences of this formula for Sturmian real numbers, for (simultaneous) Diophantine approximation and for formal power series.

Keywords: Continued fractions; Combinatorics on words; Palindromes; Sturmian sequences; Mirror formula; Folding lemma; Diophantine approximation

1. Introduction

In the present survey, a conference version of which appeared as [1], we will focus on reversals of patterns and on palindromic patterns that occur in continued fraction expansions for real numbers and for formal Laurent series with coefficients in a finite field. Our main motivation comes from the remark that various very recent, and apparently unrelated, works make use of an elementary formula for continued fractions, referred to as the *mirror formula* all along this paper (see for example [3,4,6,5,7,15,19,21,22,44,75,74] for related papers published since 2005). This leads us to review some of these results, together with older ones, and to underline the central rôle played by this formula.

The first part of the paper (Sections 4 and 5) deals with combinatorics on words. We investigate in particular some questions related to the critical exponent, to the recurrence quotient, and to the palindrome density of sequences (also called infinite words). Most of the results involve Sturmian sequences: one characterization among others of these infinite words is that they are binary codings of non-periodic trajectories on a square billiard. The continued fraction expansion of the slope of these trajectories unveils the combinatorial properties of the associated Sturmian words, which explains that the mirror formula naturally appears in this framework.

The following sections are essentially devoted to *Diophantine approximation*, which can be defined as the art of answering the question: how good an approximation of a given real number by rationals p/q as a function of q can be? Continued fractions and Diophantine approximation are of course intimately connected, since the best rational approximations to a real number are produced by truncating its continued fraction expansion. It is

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however much less known, and quite new, that continued fractions can be used in order to study some questions of simultaneous approximation (i.e., the more general problem of approximating several real numbers by rationals having the same denominators). Mainly due to the lack of a suitable multi-dimensional continued fraction algorithm, such problems are generally considered as rather difficult. We will survey some old Diophantine questions together with recent developments where continued fractions, thanks to the mirror formula, are used to provide simultaneous rational approximations for some real numbers. In this regard, Section 6 is an exception since it deals with rational approximation of (only) one real number, defined by its binary expansion. However, Section 6 is still concerned by both Diophantine approximation and the mirror formula. Section 7 addresses simultaneous approximation for a number and its square. Section 8 deals with the Littlewood conjecture. Section 9 studies the transcendence of some families of continued fractions.

2. Notations

We will use the classical notations for finite or infinite continued fractions

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_0, a_1, \ldots, a_n]$$

resp.

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}} = [a_0, a_1, \ldots, a_n, \ldots]$$

where p/q is a positive rational number, resp. α is a positive irrational real number, n is a nonnegative integer, a_0 is a nonnegative integer, and the a_i's are positive integers for $i \geq 1$. If $0 \leq k \leq n$, we denote by p_k/q_k the k-th convergent to p/q (resp. to α), i.e., $p_k/q_k := [a_0, a_1, \ldots, a_k]$. In particular, for the rational p/q we have $p/q = p_n/q_n = [a_0, a_1, \ldots, a_n]$. The sequence of denominators of the convergents to p/q (resp. to α) satisfies, for n such that $1 \leq k \leq n$, the relation $q_k = a_k q_{k-1} + q_{k-2}$, with the convention that $q_{-1} := 0$ and $q_0 := 1$.

We will also have continued fractions for formal Laurent series over a field K: in this case, p/q is a rational function (p and q are two polynomials in $K[X]$), resp. α is a Laurent series $\sum_{j \geq t} r_j X^{-j}$, n is a nonnegative integer, and the a_i's are nonzero polynomials in $K[X]$.

3. A fundamental lemma

A pleasant and useful formalism for continued fractions is the matrix formalism that we borrow from papers of van der Poorten (see, for example, [69,73]), who says that it goes back at least to [45]: we have that

$$\forall n \geq 0, \quad [a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}, \text{ with } \gcd(p_n, q_n) = 1$$

if and only if

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Taking the transposition of this equality easily yields the following lemma: