Theoretical Computer Science 516 (2014) 40–59

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Multidimensional cellular automata: closing property, quasi-expansivity, and (un)decidability issues

Alberto Dennunzio [a,*], Enrico Formenti [b,*], Michael Weiss [b]

[a] Università degli studi di Milano–Bicocca, Dipartimento di Informatica Sistemistica e Comunicazione, viale Sarca 336, 20126 Milano, Italy
[b] Université Nice–Sophia Antipolis, Laboratoire I3S, 2000 Route des Colles, 06903 Sophia Antipolis, France

ARTICLE INFO

Article history:
Received 28 February 2013
Received in revised form 30 July 2013
Accepted 5 November 2013
Communicated by B. Durand

Keywords:
Cellular automata
Symbolic dynamics
Decidability
Tiling

ABSTRACT

In this paper we study the dynamics of D-dimensional cellular automata (CA) by considering them as one-dimensional (1D) CA along some direction (slicing constructions). These constructions allow to give the D-dimensional version of important notions as 1D closing property and lift well-known one-dimensional results to the D-dimensional settings. Indeed, like in one-dimensional case, closing D-dimensional CA have jointly dense periodic orbits and biclosing D-dimensional CA are open. By the slicing constructions, we further prove that for the class of closing D-dimensional CA, surjectivity implies surjectivity on spatially periodic configurations (old standing open problem). We also deal with the decidability problem of the D-dimensional closing. By extending the Kari's construction from [31] based on tilings, we prove that the two-dimensional, and then D-dimensional, closing property is undecidable. In such a way, we add one further item to the class of dimension sensitive properties, i.e., properties that are decidable in dimension 1 and are undecidable in higher dimensions. It is well-known that there are not positively expansive CA in dimension 2 and higher. As a meaningful replacement, we introduce the notion of quasi-expansivity for D-dimensional CA which shares many global properties (in the D-dimensional settings) with the 1D positive expansivity. We also prove that for quasi-expansive D-dimensional CA the topological entropy (which is an uncomputable property for general CA) has infinite value. In a similar way as quasi-expansivity, the notions of quasi-sensitivity and quasi-almost equicontinuity are introduced and studied.

1. Introduction

Cellular automata (CA) are a well-known formal model for complex systems [12,11,34,26,8] and, at the same time, a paradigmatic model of massive parallel computation [27]. Indeed, a CA is made of a possibly infinite number of identical finite automata arranged on a regular lattice ($\mathbb{Z}^D$ or $\mathbb{Z}$ in this paper). Each automaton takes a state chosen from a set A, called the *set of states* or the *alphabet*. A *configuration* is a snapshot of all states of the automata. A *local rule* updates the state of each automaton on the basis of its current state and the ones of a finite set of neighboring automata. All automata are updated synchronously.

Historically, Von Neumann introduced CA to study formal models for cells self-reproduction. These were two-dimensional models. Also applications mainly concern two or higher dimensional CA. However, the formal study of the CA dynamical behavior concentrated essentially on the one-dimensional case except for additive CA [38,10] and a few others (see [45,22,2], for example).

* Corresponding authors.
E-mail addresses: dennunzio@disco.unimib.it (A. Dennunzio), enrico.formenti@unice.fr (E. Formenti), michael.weiss@unice.fr (M. Weiss).

0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.11.005

In this paper, in order to pursue the study of the D-dimensional case ($D \geqslant 2$), we introduce two constructions which allow to represent a D-dimensional CA as a 1D CA (*slicing constructions*, see Section 3). The basic idea is to "cut" any configuration of a CA in dimension D into slices of dimension $D-1$ defined over hyperplanes which are orthogonal to a direction $\vec{\mu}$. Hence, the former CA (in dimension D) can be seen as a new 1D CA operating on configurations made of slices (Theorem 1). The obtained 1D CA has an infinite alphabet and works on a non-compact configuration set but, by Theorem 3, the construction gives a 1D CA "approximation" with finite alphabet when the former CA is restricted to configurations which are translation invariant (the larger the translation, the finer the 1D CA approximation). In this way, well-known results of one-dimensional CA can be lifted to the D-dimensional case (results concern global properties of the D-dimensional global map on the whole D-dimensional configuration space). The main proof technique combines the slicing constructions with the fact that translation invariant configurations are dense, allowing to transfer to the original D-dimensional CA some properties of suitable 1D CA approximations. However, even if the constructions are of help for proving 1D-like results, most of their proofs are significantly different from the ones of the 1D case (cf. Theorems 8 and 12 in this paper and the corresponding 1D versions in [36] or in [5]).

The slicing constructions give a pathway on how to generalize or reshape important notions from the 1D to the D-dimensional case. In particular, the present paper generalizes the notion of closing 1D CA and introduces the concept of quasi-expansivity for D-dimensional CA, as replacement of positive expansivity (that is meaningless if $D \geqslant 2$).

The closing property is of interest in 1D symbolic dynamics since it is tightly linked to several dynamical behaviors. For instance, right (or left) closing CA have (*jointly*) *dense periodic orbits* ((J)DPO) [7], an important property of the CA dynamics which, together with transitivity and sensitivity, is also a fundamental component of the popular Devaney's definition of chaos for discrete dynamical systems [23]. In CA settings, a challenging old-standing open problem concerns DPO: is it enjoyed by all (and not only by closing) surjective CA? If the answer is affirmative then chaos in CA reduces to transitivity, due to the fact that transitive CA are both sensitive and surjective (see [1,15] for recent investigations). Moreover, the closing property is decidable. As 1D-like results, we prove that closing D-dimensional CA have JDPO (Theorem 8) and biclosing D-dimensional CA are open (Theorem 12). Furthermore, we consider an old-standing problem concerning the relation between the surjectivity of a D-dimensional CA and the surjectivity of the same CA when restricted to spatially periodic configurations [24,32]: does surjectivity imply surjectivity on spatially periodic configurations? This implication holds for 1D CA but nothing about it is known for higher dimensions (the converse is true in any dimension). We prove that the implication holds for the class of closing D-dimensional CA (Proposition 15).

Remark that 2-dimensional CA can be seen as transformations on tilings. From the fact that most properties on tilings are undecidable, one would expect that the same holds for properties of 2-dimensional CA (that is trivially true for the ones which are already undecidable in dimension 1). Indeed, Kari proved that this is the case for injectivity and surjectivity [31]. We stress that these properties are decidable in dimension 1. In Section 5, we prove that the closing property (and other properties related to it) is undecidable in the 2-dimensional, and then D-dimensional ($D \geqslant 2$), case (Theorem 16). This result corrects an error made in [16, Prop. 2] due to a wrong use of the property characterizing closing CA [16, Prop. 1]. Recalling that closing is decidable in dimension 1 (see [36]), we have just added one more item to the slowly growing collection of dimension sensitive properties (see [31,4] for other examples). Moreover, the proof technique used for Theorem 16 extends the Kari's construction [31] based on tilings. We believe that this new construction is of interest in its own.

It is well-known that for $D \geqslant 2$ there is no positively expansive D-dimensional CA [44]. This result can be restated in terms of propagation of defects, i.e., differences between pairs of configurations. Intuitively, a positively expansive CA is able to produce new defects and spread them to any direction of the cellular space. Equivalently, a CA is positively expansive if during its evolution it propagates any defect towards a finite window centered at the origin of the lattice (see Fig. 1(a) in dimension $D = 2$). If in the 1D case this can happen since there are only two directions, it no longer holds for CA over a D-dimensional lattice, due to the infinite number of possible directions. However, computer experiments show that there are many D-dimensional CA which are able to create new defects and propagate them toward a centered infinite "stripe" (individuated by a direction $\vec{\mu}$ perpendicular to the stripe, see Fig. 1(b) in dimension $D = 2$). In some sense, these CA "seem" expansive. To capture this intuition we introduce the directional concept of quasi-expansivity ($\vec{\mu}$-expansivity) and we prove that it shares many global properties with the 1D CA positive expansivity. In particular, quasi-expansivity implies the following global D-dimensional CA properties: topological mixing, JDPO, openness, and surjectivity (Theorem 20 and Corollary 19). Moreover, similarly to the 1D case, bipermutative CA (i.e., the ones with a local rule which is permutative in two suitable positions) form a subclass of quasi-expansive CA. Furthermore, quasi-expansive CA turn out to be another class of CA with infinite topological entropy (Theorem 31). This value is obtained as infinite limit of growing finite entropies of positively expansive 1D CA approximating the D-dimensional CA. These results support our idea that quasi-expansivity is the good notion for studying "this kind" of dynamics in dimension 2 and higher and prove that a directional concept is not a limitation for studying D-dimensional CA. They also motivate us to consider other D-dimensional directional properties. By a result in [45], the classical dichotomy between sensitive and almost equicontinuous 1D CA is no longer true in dimension 2 or higher. In this paper, we prove that a dichotomy still holds (Proposition 32) if the directional notions of $\vec{\mu}$-sensitivity and $\vec{\mu}$-almost equicontinuity are considered. Furthermore, these properties turn out to be undecidable (Proposition 33).

The paper is structured as follows. The next section recalls basic notions and some known results about CA, discrete dynamical systems, and tilings. Section 3 presents the slicing constructions. Section 4 contains the results on the dynamics of closing CA. Section 5 is devoted to the undecidability of closing. In Section 6 we deal with quasi-expansivity. Partial versions of some results from this paper were presented in [16] and [21].