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Approximability of the vertex cover problem in power-law graphs

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ABSTRACT

In this paper we construct an approximation algorithm for the Minimum Vertex Cover (Min-VC) problem with an expected approximation ratio of $2 - \frac{\zeta(\beta)-1-\frac{1}{2^\beta}}{2^\beta \zeta(\beta-1)\zeta(\beta)}$ for *random power-law graphs* in the $P(\alpha, \beta)$ model due to Aiello et al. Here $\zeta(\beta)$ is the Riemann zeta function of β. We obtain this result by combining the Nemhauser and Trotter approach for Min-VC with a new deterministic rounding procedure which achieves an approximation ratio of 3/2 on a subset of low degree vertices for which the expected contribution to the cost of the associated linear program is sufficiently large.

1. Introduction

In recent years topological analyses have been applied to a variety of real world graphs such as the World-Wide Web, the Internet, Collaboration and Social Networks, Protein Interaction Networks and other large-scale graphs of biological systems. Typical statistical parameters such as the *diameter, robustness, clustering coefficient* and *degree distribution* have been measured and compared to the expected values of these parameters in *uniform* random graph models such as the classical $G(n, p)$ model due to Erdős and Rényi [1]. It turned out that the real world graphs are significantly different from the uniform random models with respect to these statistical and topological properties. In subsequent studies the aim was to describe the properties of real world networks mathematically and to propose new models in order to meet these conditions.

As of 1999 Kumar et al. [2,3], Kleinberg et al. [4,5] and Faloutsos, Faloutsos and Faloutsos [6,7] measured the degree distribution of the World-Wide Web and independently observed that it is well approximated by a *power-law distribution*, i.e. the number of nodes y_i of a given degree i is proportional to $i^{-\beta}$ where $\beta > 0$. This was later verified for a large number of existing real-world networks such as protein–protein interactions, gene regulatory networks, peer-to-peer networks, mobile call networks and social networks [8–11]. In order to analyze these graphs, some research has been directed towards finding suitable models for describing structural properties quantitatively and qualitatively. A number of *power-law graph* (PLG) models have been proposed, such as the Barabási–Albert model of *Preferential Attachment* [12], the *LCD model* [13], the *Buckley–Osthus model* [14], the *Cooper–Frieze model* [15] and the *Copying model* due to Kumar et al. [3]. All these models describe a *random growth process* starting from a small seed graph and yielding—besides other features—a power-law degree distribution in the limit.

A different approach is to take a power-law degree sequence as input and to generate a graph instance with this distribution in a random fashion. Among the most widely known models of this kind is the *ACL model* due to Aiello, Chung and Lu [16]. Here, the number y_i of vertices of degree i is roughly given by $y_i \approx e^\alpha / i^\beta$, where e^α is a normalization constant which

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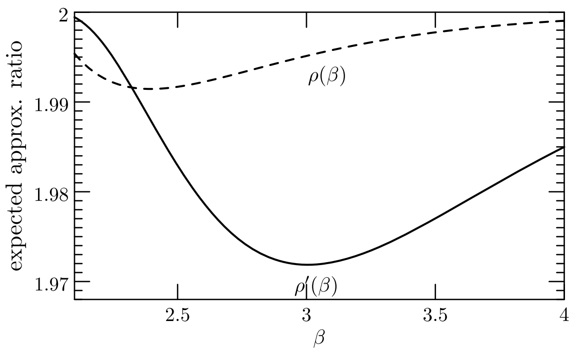

Fig. 1. Comparison of first (- - -) and second (——) analysis for $\beta > 2$. Particularly, $\rho'(\beta)$ yields an improvement over $\rho(\beta)$ for $\beta > 2.322$.

determines the size of the graph. While this model is potentially less accurate than the detailed description of a growth process of an evolving graph, it has the advantage of being robust and general, i.e., structural properties that are true in this model will be true for the majority of graphs with the given degree sequence. All of the above models are well motivated and there exists a large body of literature on mathematical foundations, properties and applications [12,17,13,11,18]. In this paper, we focus on the ACL model for random PLG which we will refer to as the *$P(\alpha, \beta)$ model*.

Apart from having certain structural properties, such as a power-law degree distribution, high clustering coefficient, small-world characteristics and self-similarity, there exists practical evidence that combinatorial optimization in PLG is easier than in general graphs [19,20,11,21]. Contrasting this Ferrante et al. [22] and Shen et al. [23] studied the approximation hardness of certain optimization problems in *combinatorial power-law graphs* and showed NP-hardness and APX-hardness of classical problems such as Minimum Vertex Cover (Min-VC), Maximum Independent Set (Max-IS) and Minimum Dominating Set (Min-DS). In this paper we study the approximability of the Min-VC problem in the random power-law graph model of Aiello et al. [16].

The Min-VC problem is one of the most well-studied problems in combinatorial optimization. A *vertex cover* of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that each edge $e = \{u, v\}$ of G has at least one endpoint in C. The Min-VC problem is the problem of finding a vertex cover of minimum cardinality in a graph. The problem is known to be NP-complete due to Karp's original proof [24] and APX-complete [25]. Moreover, it cannot be approximated within a factor of 1.3606 [26], unless P = NP, and is inapproximable within $2 - \epsilon$ for any $\epsilon > 0$ as long as the Unique Games Conjecture (UGC) holds true [27]. Here, we show that the Min-VC problem can be approximated with an expected approximation ratio < 2 in random power-law graphs in the $P(\alpha, \beta)$ model (which will be properly described in Section 2.1):

Theorem 1. *For $\beta > 2$, there exists a polynomial time algorithm which approximates the* Minimum Vertex Cover *problem in random power-law graphs in the $P(\alpha, \beta)$ model with an expected approximation ratio of*

$$\rho = 2 - \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^\beta \zeta(\beta - 1)\zeta(\beta)},$$

where $\zeta(\beta) = \sum_{j=1}^{\infty} \frac{1}{j^\beta}$ is the Riemann zeta function.

We also give a refined analysis which yields an even better approximation ratio for the case of the parameter $\beta > 2.322$.

Theorem 2. *For $\beta > 2$, the* Minimum Vertex Cover *problem in random power-law graphs in the $P(\alpha, \beta)$ model can be approximated with expected asymptotic approximation ratio of*

$$\rho' = 2 - \frac{(\zeta(\beta) - 1 - \frac{1}{2^\beta})\zeta(\beta - 1)}{\zeta(\beta - 1)\zeta(\beta)}\left[1 - \left(\frac{\zeta(\beta - 1) - (1 + \frac{1}{2^{\beta-1}})}{\zeta(\beta - 1)}\right)^3\right].$$

In Fig. 1 the two upper bounds ρ and ρ' of Theorem 1 and Theorem 2 are shown as functions of the parameter β for $\beta > 2$.

The paper is organized as follows. In Section 2.1 we describe the $P(\alpha, \beta)$ model for power-law graphs, describe the random generation process of graphs $G_{\alpha,\beta} \in \mathcal{G}_{\alpha,\beta}$ and give a formal description of the model parameters. In Section 2.2 we give some background on the Min-VC problem and briefly describe the *half-integral solution* method proposed by Nemhauser and Trotter. Section 3 presents our new approximation algorithm for Min-VC in power-law graphs. This algorithm basically consists of a deterministic rounding procedure on a half-integral solution for Min-VC. In Section 3.1 we show that this rounding procedure yields an approximation ratio of 3/2 in the subgraph induced by the low degree vertices of the power-law graph and a 2-approximation in the residual graph. In Section 3.2 we construct upper and lower bounds on the expected size of